

ENUMERATIONS OF LOZENGE TILINGS, LATTICE PATHS, AND PERFECT MATCHINGS AND THE WEAK LEFSCHETZ PROPERTY

DAVID COOK II* AND UWE NAGEL

ABSTRACT. MacMahon enumerated the plane partitions in an $a \times b \times c$ box. These are in bijection to lozenge tilings of a hexagon, to certain perfect matchings, and to families of non-intersecting lattice paths. In this work we consider more general regions, called triangular regions, and establish signed versions of the latter three bijections. Indeed, we use perfect matchings and families of non-intersecting lattice paths to define two signs of a lozenge tiling. A combinatorial argument involving a new method, called resolution of a puncture, then shows that the signs are in fact equivalent. This provides in particular two different determinantal enumerations of these families. These results are then applied to study the weak Lefschetz property of Artinian quotients by monomial ideals of a three-dimensional polynomial ring. We establish sufficient conditions guaranteeing the weak Lefschetz property as well as the semistability of the syzygy bundle of the ideal, classify the type two algebras with the weak Lefschetz property, and study monomial almost complete intersections in depth. Furthermore, we develop a general method that often associates to an algebra that fails the weak Lefschetz property a toric surface that satisfies a Laplace equation. We also present examples of toric varieties that satisfy arbitrarily many Laplace equations. Our combinatorial methods allow us to address the dependence on the characteristic of the base field for many of our results.

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* Corresponding author.

1. INTRODUCTION

A plane partition is a rectangular array of nonnegative integers such that the entries in each row and each column are weakly decreasing. It is in an $a \times b \times c$ box if the array has a rows, b columns, and all entries are at most c . MacMahon [41] showed that the number of plane partitions in an $a \times b \times c$ box is

$$\frac{\mathcal{H}(a)\mathcal{H}(b)\mathcal{H}(c)\mathcal{H}(a+b+c)}{\mathcal{H}(a+b)\mathcal{H}(a+c)\mathcal{H}(b+c)},$$

where a , b , and c are nonnegative integers and $\mathcal{H}(n) := \prod_{i=0}^{n-1} i!$ is the *hyperfactorial* of n . It is well-known that this result can be interpreted as counting the number of lozenge tilings of a hexagon with side lengths a, b, c . Think of a plane partition as a stack of unit cubes, where the number of stacked cubes in position (i, j) is given by the corresponding entry in the array, as illustrated in Figure 1.1.

$$\begin{array}{cccccc} 3 & 3 & 2 & 2 & 2 & 1 \\ 3 & 2 & 2 & 1 & 0 & 0 \end{array}$$

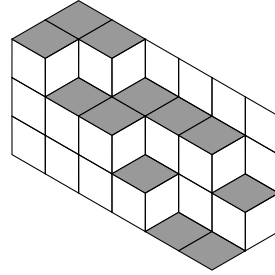


FIGURE 1.1. A $2 \times 6 \times 3$ plane partition and the corresponding stack of cubes. The grey lozenges are the tops of the boxes.

The projection of the stack to the plane with normal vector $(1, 1, 1)$ gives a lozenge tiling of a hexagon with side lengths a, b, c (see Figure 1.2). Here the hexagon is considered as a union of equilateral triangles of side length one, and a lozenge is obtained by gluing together two such triangles along a shared edge.

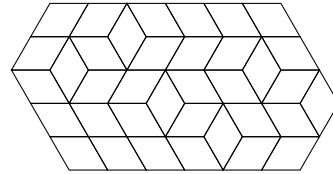
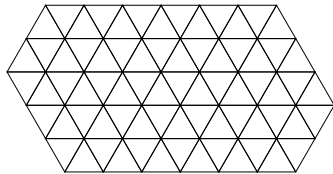


FIGURE 1.2. A $2 \times 6 \times 3$ hexagon and the lozenge tiling associated to the plane partition in Figure 1.1.

In this work we view the above hexagon as a subregion of a triangular region \mathcal{T}_d , which is an equilateral triangle of side length d subdivided by equilateral triangles of side length one. See Figure 1.3 for an illustration.

The hexagon with side lengths a, b, c is obtained by removing triangles of side lengths a, b , and c at the vertices of \mathcal{T}_d , where $d = a + b + c$. We refer to the removed upward-pointing triangles as *punctures*. More generally, we consider subregions $T \subset \mathcal{T}$ that arise from \mathcal{T} by removing upward-pointing triangles, each of them being a union of unit triangles. The punctures, that is, the removed upward-pointing triangles may overlap (see Figure 1.4). We


 FIGURE 1.3. The triangular regions \mathcal{T}_3 and \mathcal{T}_4 .

call the resulting subregions of \mathcal{T} *triangular subregions*. Such a region is said to be *balanced* if it contains as many upward-pointing unit triangles as down-pointing unit triangles. For example, hexagonal subregions are balanced. Lozenge tilings of triangular subregions have been studied in several areas. For example, they are used in statistical mechanics for modeling bonds in dimers (see, e.g., [31]) or in statistical mechanics when studying phase transitions (see, e.g., [12]).



FIGURE 1.4. Two triangular subregions.

If a triangular subregion T is a hexagon with side lengths a, b, c , then the plane partitions in an $a \times b \times c$ box are not only in bijection to lozenge tilings of T , but also to perfect matchings determined by T as well as to families of non-intersecting lattice paths in T (see, e.g., [53]). Moreover, all these objects are enumerated by a determinant of an integer matrix. For more general balanced triangular subregions, the latter three bijections remain true, whereas the bijection to plane partitions is lost.

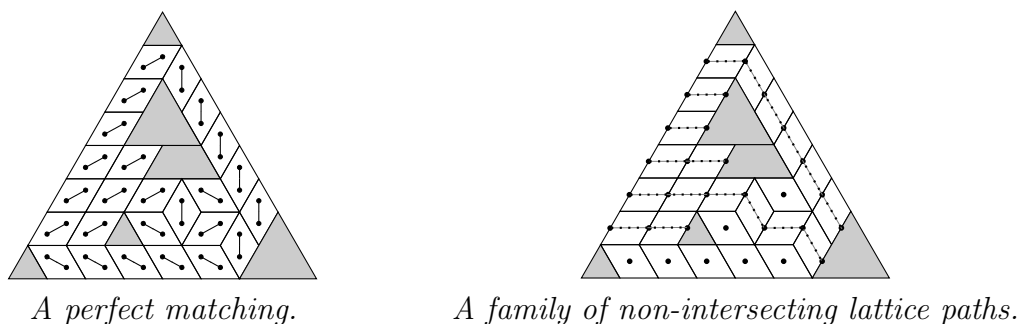


FIGURE 1.5. Bijections to lozenge tilings.

Here we establish a signed version of these bijections. Introducing suitable signs, one of our main results says that, for each balanced triangular subregion T , there is a bijection between the signed perfect matchings and the signed families of non-intersecting lattice paths. This is achieved via the links to lozenge tilings.

Indeed, using the theory pioneered by Gessel and Viennot [22], Lindström [39], Stembridge [59], and Krattenthaler [32], the sets of signed families of non-intersecting lattice paths in T can be enumerated by the determinant of a matrix $N(T)$ whose entries are binomial coefficients, once a suitable sign is assigned to each such family. We define this sign as the *lattice path sign* of the corresponding lozenge tiling of the region T .

The perfect matchings determined by T can be enumerated by the permanent of a zero-one matrix $Z(T)$ that is the bi-adjacency matrix of a bipartite graph. This suggests to introduce the sign of a perfect matching such that the signed perfect matchings are enumerated by the determinant of $Z(T)$. We call this sign the *perfect matching sign* of the lozenge tiling that corresponds to the perfect matching. Typically, the matrix $N(T)$ is much smaller than the matrix $Z(T)$. However, the entries of $N(T)$ can be much bigger than one. Nevertheless, a delicate combinatorial argument shows that the perfect matching sign and the lattice path sign are equivalent, and thus (see Theorem 5.17)

$$|\det Z(T)| = |\det N(T)|.$$

The proof also reveals instances where the absolute value of $\det Z(T)$ is equal to the permanent of $Z(T)$ (see Proposition 6.15). This includes hexagonal regions, for which the result is well-known.

The above results allow us to obtain explicit enumerations in many new instances. They also suggest several intriguing conjectures.

Another starting point and motivation for our investigations has been the problem of deciding the presence of the Lefschetz properties. A standard graded Artinian algebra A over a field K is said to have the *weak Lefschetz property* if there is a linear form $\ell \in A$ such that the multiplication map $\times \ell : [A]_i \rightarrow [A]_{i+1}$ has maximal rank for all i (i.e., it is injective or surjective). The algebra A has the *strong Lefschetz property* if $\times \ell^d : [A]_i \rightarrow [A]_{i+d}$ has maximal rank for all i and d . The names are motivated by the conclusion of the Hard Lefschetz Theorem on the cohomology ring of a compact Kähler manifold. Many algebras are expected to have the Lefschetz properties. However, establishing this fact is often very challenging.

The Lefschetz properties play a crucial role in the proof of the so-called g -Theorem. It characterises the face vectors of simplicial polytopes, confirming a conjecture of McMullen. The sufficiency of McMullen's condition was shown by Billera and Lee [2] by constructing suitable polytopes. Stanley [57] established the necessity of the conditions by using the Hard Lefschetz Theorem to show that the Stanley-Reisner ring of a simplicial polytope modulo a general linear system of parameters has the strong Lefschetz property. It has been a longstanding conjecture whether McMullen's conditions also characterise the face vectors of all triangulations of a sphere. This conjecture would follow if one can show that the Stanley-Reisner ring of such a triangulation modulo a general linear system of parameters has the weak Lefschetz property. The algebraic g -Conjecture posits that this algebra even has the strong Lefschetz property. If true, this would imply strong restrictions on the face vectors of all orientable K -homology manifolds (see [49] and [50]). Although there has been a flurry of papers studying the Lefschetz properties in the last decade (see, e.g., [4, 8, 9, 10, 18, 26, 27, 36, 37, 38, 43, 46]), we currently seem far from being able to decide the above conjectures. Indeed, the need for new methods has led us to consider lozenge tilings, perfect matchings, and families of non-intersecting lattice paths. We use this approach to establish new results about the presence or the absence of the weak Lefschetz property of

quotients of a polynomial ring $R = K[x, y, z]$ by a monomial ideal I that contains powers of each of the variables x , y , and z .

In the case where the ideal I has only three generators, the powers of the variables, the algebra R/I has the Lefschetz properties if the base field has characteristic zero (see [56, 55, 62, 17]). In this case the algebra R/I has Cohen-Macaulay type one. We extend this result in several directions.

First, one of the main results in [3] says that the monomial algebras R/I of type two that are also level have the weak Lefschetz property if K has characteristic zero. Examples show that this may fail if one drops the level assumption or if K has positive characteristic. However, the intricate proof in the level case in [3] did not give any insight when such failures occur. We resolve this by completely classifying all type two algebras that have the weak Lefschetz property if the characteristic is zero or large enough (Theorem 9.2 and Proposition 9.9).

Second, we consider the case where the ideal I is an almost complete intersection, that is, I is minimally generated by four monomials. We decide the presence of the weak Lefschetz property in a broad range of cases, adding, for example, new evidence to a conjecture in [45]. In particular, we show that the weak Lefschetz property may fail in at most one degree, that is, the multiplication by a general linear form $[R/I]_{j-1} \rightarrow [R/I]_j$ has maximal rank for all but at most one integer j (see Theorem 10.9).

Furthermore, we establish the weak Lefschetz property for various other infinite classes of algebras R/I , where the ideal I can have arbitrarily many generators.

If an algebra that is expected to have the weak Lefschetz property actually fails to have it, this is often of interest too. A projective variety is said to satisfy a Laplace equation of order s if its s -th osculating space at a general (smooth) point has smaller dimension than expected. Togliatti [61] started investigating such varieties and obtained the first classification results. Very recently, Mezzetti, Miró Roig, and Ottaviani [43] showed that the existence of Laplace equations is closely related to the failure of the weak Lefschetz property. Using this, we prove that *every* Artinian monomial ideal $I \subset R$ such that R/I fails injectivity in degree $d - 1$ as predicted by the weak Lefschetz property, that is, the multiplication map $\times \ell : [R/I]_{d-1} \rightarrow [R/I]_d$ is not injective and $0 < \dim_K [R/I]_{d-1} \leq \dim_K [R/I]_d$, gives rise to a toric surface satisfying a Laplace equation of order $d - 1$ (see Theorem 11.10). Furthermore, we use our approach via lozenge tilings to construct toric surfaces that satisfy any desired number of independent Laplace equations of order $d - 1$ whenever d is sufficiently large (Corollary 11.17).

The key to relating results on lozenge tilings to the study of the Lefschetz properties is to label the unit triangles in a triangular region by monomials. This allows us to translate properties of a monomial ideal I into properties of its associated triangular subregions $T_d(I) \subset \mathcal{T}_d$. This is described in Section 2.

In Section 3 we establish sufficient and necessary conditions for a balanced triangular subregion to be tileable (see, e.g., Theorem 3.2). Our arguments also give an algorithm for constructing a tiling of a triangular subregion if any such tiling exists.

It turns out that the tileability of a triangular subregion $T_d(I)$ is related to the semistability of the syzygy bundle of the ideal I . This is established in Section 4 (see Theorem 4.5).

Key results of our approach are developed in Section 5. First, in Subsection 5.1 we recall that every non-empty subregion T of \mathcal{T}_d corresponds to a bipartite graph. We use this bijection to define the bi-adjacency matrix $Z(T)$ and to introduce the perfect matching sign

of a lozenge tiling. Second, we consider families of non-intersecting lattice path in T and introduce the lattice path matrix $N(T)$ as well as the lattice path sign of a lozenge tiling in Subsection 5.2. In order to compare the perfect matching and the lattice path sign of lozenge tilings we introduce a new combinatorial construction that we call *resolution of a puncture* (see Subsection 5.3). Roughly speaking, it replaces a triangular subregion with a fixed lozenge tiling by a larger triangular subregion with a compatible lozenge tiling and one puncture less. Carefully analyzing the change of sign under resolutions of punctures and using induction on the number of punctures of a given region, we establish that, for each balanced triangular subregion, the two defined signs of a lozenge tiling are in fact equivalent, and thus, $|\det N(T)| = |\det Z(T)|$. This results allows us to move freely between signed perfect matchings and families of non-intersecting lattice paths.

In Section 6 we use this interplay and MacMahon's enumeration of plane partitions to establish various explicit enumerations. We also give sufficient conditions that guarantee that all lozenge tilings of a triangular subregion have the same sign (see Proposition 6.15). In this case, the permanent of $Z(T)$, which gives the total number of perfect matchings determined by T , is equal to $|\det Z(T)|$.

The special case of a mirror symmetric region is considered in Section 7. Using a result by Ciucu [12], we provide some explicit enumerations of signed perfect matchings (see Theorems 7.8 and 7.11). We also offer a conjecture (Conjecture 7.6) on the regularity of the bi-adjacency matrix of a mirror symmetric region and provide evidence for it.

In the remainder of this work we apply the results on lozenge tilings to study the Lefschetz properties. In Section 8 we first present some general tools for establishing the weak Lefschetz property. Then we show that, for an Artinian monomial ideal $I \subset R$, the rank of the multiplication $\times \ell : [R/I]_{d-2} \rightarrow [R/I]_{d-1}$ by a general linear form ℓ is governed by the rank of the bi-adjacency matrix $Z(T)$ (see Proposition 8.10) and the rank of the lattice path matrix $N(T)$ (see Proposition 8.15) of the region $T = T_d(I)$, respectively. Since these are integer matrices, it follows that in the case where these matrices have maximal rank, the prime numbers dividing all their maximal minors are the positive characteristics of the base field K for which R/I fails to have the weak Lefschetz property. We first draw consequences to monomial complete intersections and then establish some sufficient conditions on a monomial ideal I such that R/I has the weak Lefschetz property and the syzygy bundle of I is semistable in characteristic zero (see Theorem 8.28).

Algebras of type two are investigated in Section 9. Theorem 9.2 gives the mentioned classification of such algebras with the weak Lefschetz property in characteristic zero, extending the earlier result for level algebras in [3].

In Section 10 we consider an Artinian monomial ideal $I \subset R$ with four minimal generators. Our results on the weak Lefschetz property of R/I are summarised in Theorem 10.9. In particular, they provide further evidence for a conjecture in [45], which concerns the case where R/I is also level. Furthermore, we determine the generic splitting type of the syzygy bundle of I in all cases but one (see Propositions 10.19 and 10.21). In the remaining case we show that determining the generic splitting type is equivalent to deciding whether R/I has the weak Lefschetz property (see Theorem 10.23).

The results on varieties satisfying Laplace equations are established in Section 11. They are based on Proposition 11.5. It says that in order to decide whether the bi-adjacency matrix $Z(T)$ of a triangular region has maximal rank, it is enough to decide the same problem for a modification \hat{T} of T whose punctures all have side length one. In Subsection 11.3 we also

give examples of balanced triangular subregions T such that its bi-adjacency matrix $Z(T)$ is regular and $\det Z(T)$ has remarkably large prime divisors. In fact, assuming a number-theoretic conjecture by Bouniakowsky, we exhibit triangular subregions $T_d \subset \mathcal{T}_d$ such that $\det Z(T_d) \neq 0$ has prime divisors of the order d^2 .

We conclude by discussing some open problems that are motivated by this work in Section 12.

2. IDEALS AND TRIANGULAR REGIONS: A DICTIONARY

In this section, we introduce a correspondence between monomial ideals and triangular regions. We define a few helpful terms for triangular regions which allow us to interpret properties of monomial ideals as properties of triangular regions.

2.1. Monomial ideals in three variables.

Let $R = K[x, y, z]$ be the standard graded polynomial ring over the field K , i.e., $\deg x = \deg y = \deg z = 1$. Unless specified otherwise, K is always an arbitrary field.

Let I be a monomial ideal of R . As R/I is standard graded, we can decompose it into finite vector spaces called the *homogeneous components (of R/I) of degree d* , denoted $[R/I]_d$. For $d \in \mathbb{Z}$, the monomials of R of degree d that are *not* in I form a K -basis of $[R/I]_d$.

2.2. The triangular region in degree d .

Let $d \geq 1$ be an integer. Consider an equilateral triangle of side length d that is composed of $\binom{d}{2}$ downward-pointing (∇) and $\binom{d+1}{2}$ upward-pointing (\triangle) equilateral unit triangles. We label the downward- and upward-pointing unit triangles by the monomials in $[R]_{d-2}$ and $[R]_{d-1}$, respectively, as follows: place x^{d-1} at the top, y^{d-1} at the bottom-left, and z^{d-1} at the bottom-right, and continue labeling such that, for each pair of an upward- and a downward-pointing triangle that share an edge, the label of the upward-pointing triangle is obtained from the label of the downward-pointing triangle by multiplying with a variable. The resulting labeled triangular region is the *triangular region (of R) in degree d* and is denoted \mathcal{T}_d . See Figure 2.1 for an illustration.

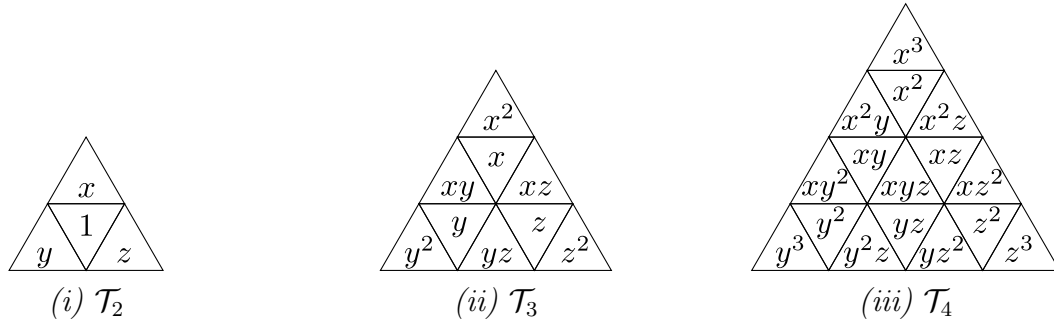


FIGURE 2.1. Some triangular regions \mathcal{T}_d .

Throughout this manuscript we order the monomials of R with the *graded reverse-lexicographic order*, that is, $x^a y^b z^c > x^p y^q z^r$ if either $a + b + c > p + q + r$ or $a + b + c = p + q + r$ and the *last* non-zero entry in $(a - p, b - q, c - r)$ is *negative*. For example, in degree 3,

$$x^3 > x^2y > xy^2 > y^3 > x^2z > xyz > y^2z > xz^2 > yz^2 > z^3.$$

Thus in \mathcal{T}_4 , see Figure 2.1(iii), the upward-pointing triangles are ordered starting at the top and moving down-left in lines parallel to the upper-left edge.

We generalise this construction to quotients by monomial ideals. Let I be a monomial ideal of R . The *triangular region (of R/I) in degree d* , denoted by $T_d(I)$, is the part of \mathcal{T}_d that is obtained after removing the triangles labeled by monomials in I . Note that the labels of the downward- and upward-pointing triangles in $T_d(I)$ form K -bases of $[R/I]_{d-2}$ and $[R/I]_{d-1}$, respectively. It is sometimes more convenient to illustrate such regions with the removed triangles darkly shaded instead of being removed; both illustration methods will be used throughout this manuscript. See Figure 2.2 for an example.

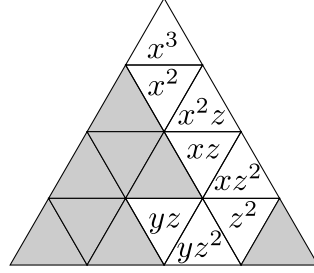


FIGURE 2.2. The triangular region $T_4(xy, y^2, z^3)$.

Notice that the regions missing from \mathcal{T}_d in $T_d(I)$ can be viewed as a union of (possibly overlapping) upward-pointing triangles of various side lengths that include the upward- and downward-pointing triangles inside them. Each of these upward-pointing triangles corresponds to a minimal generator of I that has, necessarily, degree at most $d - 1$. We can alternatively construct $T_d(I)$ from \mathcal{T}_d by removing, for each minimal generator $x^a y^b z^c$ of I of degree at most $d - 1$, the *puncture associated to $x^a y^b z^c$* which is an upward-pointing equilateral triangle of side length $d - (a + b + c)$ located a triangles from the bottom, b triangles from the upper-right edge, and c triangles from the upper-left edge. See Figure 2.3 for an example. We call $d - (a + b + c)$ the *side length of the puncture associated to $x^a y^b z^c$* , regardless of possible overlaps with other punctures in $T_d(I)$.

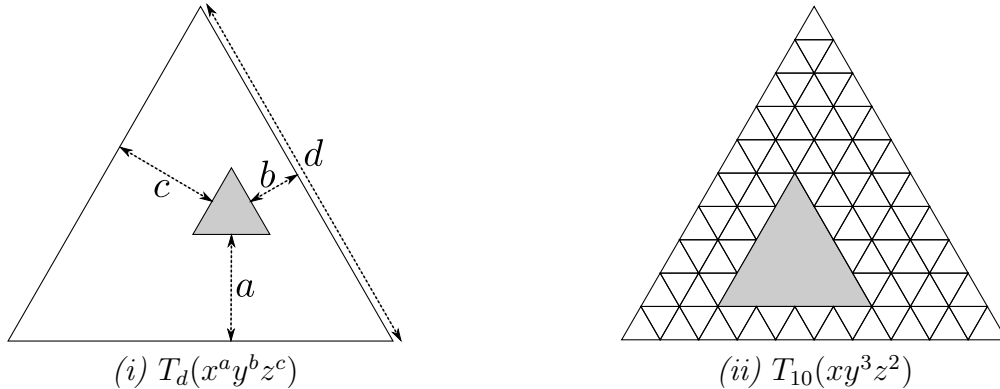


FIGURE 2.3. $T_d(I)$ as constructed by removing punctures.

We say that two punctures *overlap* if they share at least an edge. Two punctures are said to be *touching* if they share precisely a vertex.

2.3. The Hilbert function and $T_d(I)$.

Let I be a monomial ideal of R . Recall that each component $[R/I]_d$ is a finite dimensional vector space. The *Hilbert function* of R/I is the function $h_{R/I} : \mathbb{Z} \rightarrow \mathbb{Z}$, where $h(d) := h_{R/I}(d) := \dim_K[R/I]_d$. By construction, $T := T_d(I)$ has $h(d-2)$ downward-pointing triangles and $h(d-1)$ upward-pointing triangles. Notice also that $h(d)$ is the number of vertices in $T_d(I)$. Later it will become important to distinguish whether $h(d-2)$ and $h(d-1)$ are equal. We say T is *balanced* if $h(d-2) = h(d-1)$, and otherwise we say T is *unbalanced*. Moreover, for T unbalanced, if $h(d-2) < h(d-1)$, then we say T is ∇ -heavy, and otherwise we say T is \triangle -heavy.

2.4. Socle elements.

Let I be a monomial ideal of R . The quotient ring R/I or simply I is called *Artinian* if R/I is a finite K -vector space. In the language of triangular regions, this translates as R/I is Artinian if and only if $T_d(I)$ has a puncture in each corner of \mathcal{T}_d for some d .

The *socle* of R/I is the annihilator of $\mathfrak{m} = (x, y, z)$, the homogeneous maximal ideal of R , that is, $\text{soc } R/I = \{f \in R/I \mid fx = fy = fz = 0\}$. As I is a monomial ideal, $\text{soc } R/I$ can be generated by monomials. The monomials $m \in \text{soc } R/I$ of degree $m-2$ are precisely those that are the center of “triads” in $T_d(I)$. See Figure 2.4 for an illustration of such a triad.

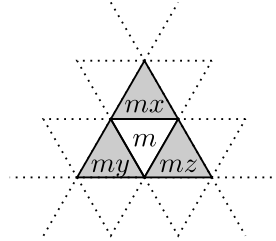


FIGURE 2.4. A triad around the socle element m .

It is often important to determine the minimal degree, or bounds thereon, of socle elements of R/I . If the punctures of $T_d(I)$ corresponding to the minimal generators of I do not overlap, then the minimal degree of a socle element is $d-2$ provided $T_d(I)$ contains a triad, otherwise the minimal degree of a socle element of R/I is at least $d-1$. On the other hand, if $T_d(I)$ has overlapping punctures, then the degrees of socle generators cannot be immediately estimated.

If R/I is Artinian, then the least degree j such that $[R/I]_j \neq 0$ is called the *socle degree* or Castelnuovo-Mumford regularity of R/I . The *type* of R/I is the K -dimension of $\text{soc } R/I$. Notice that $[R/I]_e \subset [\text{soc } R/I]_e$ if e is the socle degree of R/I . Further, R/I is said to be *level* if its socle is concentrated in one degree, i.e., in its socle degree.

2.5. Greatest common divisors.

Let I be a monomial ideal of R minimally generated by the monomials f_1, \dots, f_n . Without loss of generality, assume f_1, \dots, f_m have degrees bounded above by $d-1$. Set $g = \gcd\{f_1, \dots, f_m\}$. In \mathcal{T}_d , the puncture associated to g is exactly the smallest upward-pointing triangle that contains the punctures associated to f_1, \dots, f_m . See Figure 2.5 for an example.

The monomial ideal $J = (I, g)$ is minimally generated by g and f_{m+1}, \dots, f_n . Its triangular region $T_d(J)$ is obtained from $T_d(I)$ by replacing the punctures associated to f_1, \dots, f_m by their smallest containing puncture in $T_d(I)$. This replacing operation can be reversed. A

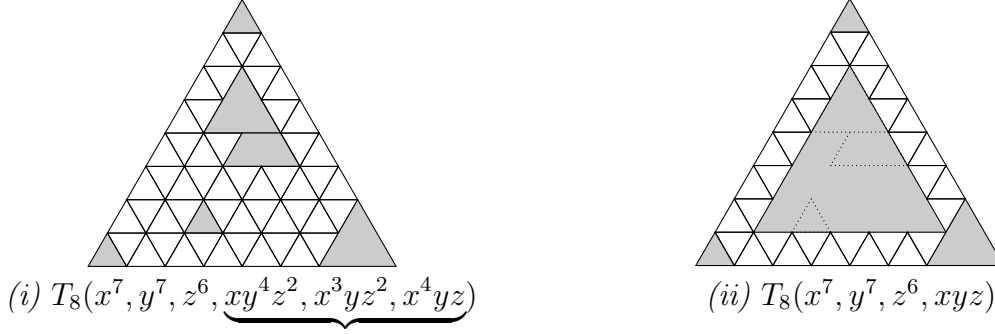


FIGURE 2.5. The greatest common divisor is associated with the minimal containing puncture.

given puncture can be broken into smaller punctures whose smallest containing puncture is the primal puncture. This corresponds to replacing a minimal generator in a monomial ideal by several multiples whose greatest common divisor is the generator.

Observe that different monomial ideals can determine the same monomial region of \mathcal{T}_d . Consider, for example, $I_1 = (x^5, y^5, z^5, xyz^2, xy^2z, x^2yz)$ and $I_2 = (x^5, y^5, z^5, xyz)$. Then $T_6(I_1) = T_6(I_2)$. However, given a triangular region $T = T_d(I)$, there is a unique largest ideal J that is generated by monomials whose degrees are bounded above by $d - 1$ and that satisfies $T = T_d(J)$. We call $J(T) := J$ the *monomial ideal of the triangular region T* . In the example, $I_2 = J(T_6(I_1))$.

Recall that each monomial of degree less than d determines a puncture in \mathcal{T}_d . Thus, the punctures of a monomial ideal $I \subset R$ in \mathcal{T}_d correspond to the minimal generators of I of degree less than d . However, the punctures of the triangular region $T = T_d(I)$ correspond to the minimal generators of $J(T)$. In the above example, I_1 determines six punctures in $T = T_6(I_1)$, but the region T has four punctures.

3. TILINGS WITH LOZENGES

In this section, we consider the question of tileability of a triangular region. Here we use monomial ideals merely as a bookkeeping tool in order to describe the considered regions. If possible we want to tile such a region by lozenges. A *lozenge* is a union of two unit equilateral triangles glued together along a shared edge, i.e., a rhombus with unit side lengths and angles of 60° and 120° . Lozenges are also called calissons and diamonds in the literature.

Fix a positive integer d and consider the triangular region \mathcal{T}_d as a union of unit triangles. Thus a *subregion* $T \subset \mathcal{T}_d$ is a subset of such triangles. We retain their labels. As above, we say that a subregion T is ∇ -heavy, \triangle -heavy, or *balanced* if there are more downward pointing than upward pointing triangles or less, or if their numbers are the same, respectively. A subregion is *tileable* if either it is empty or there exists a tiling of the region by lozenges such that every triangle is part of exactly one lozenge. See Figure 3.1. Since a lozenge in \mathcal{T}_d is the union of a downward-pointing and an upward-pointing triangle, and every triangle is part of exactly one lozenge, a tileable subregion is necessarily balanced.

Let $T \subset \mathcal{T}_d$ be any subregion. Given a monomial $x^a y^b z^c$ with degree less than d , the *monomial subregion* of T associated to $x^a y^b z^c$ is the part of T contained in the triangle a

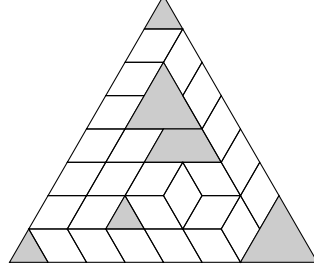


FIGURE 3.1. One of 13 tilings of $T_8(x^7, y^7, z^6, xy^4z^2, x^3yz^2, x^4yz)$ (see Figure 2.5(i)).

units from the bottom edge, b units from the upper-right edge, and c units from the upper-left edge. In other words, this monomial subregion consists of the triangles that are in T and the puncture associated to the monomial $x^a y^b z^c$. See Figure 3.2 for an example.

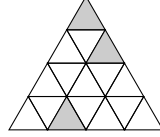


FIGURE 3.2. The monomial subregion of $T_8(x^7, y^7, z^6, xy^4z^2, x^3yz^2, x^4yz)$ (see Figure 2.5(i)) associated to xy^2z .

Replacing a tileable monomial subregion by a puncture of the same size does not alter tileability.

Lemma 3.1. *Let $T \subset \mathcal{T}_d$ be any subregion. If the monomial subregion U of T associated to $x^a y^b z^c$ is tileable, then T is tileable if and only if $T \setminus U$ is tileable.*

Moreover, each tiling of T is obtained by combining a tiling of $T \setminus U$ and a tiling of U .

Proof. Suppose T is tileable, and let τ be a tiling of T . If a tile in τ contains a downward-pointing triangle of U , then the upward-pointing triangle of this tile also is in U . Hence, if any lozenge in τ contains exactly one triangle of U , then it must be an upward-pointing triangle. Since U is balanced, this would leave U with a downward-pointing triangle that is not part of any tile, a contradiction. It follows that τ induces a tiling of U , and thus $T \setminus U$ is tileable.

Conversely, if $T \setminus U$ is tileable, then a tiling of $T \setminus U$ and a tiling of U combine to a tiling of T . \square

Let $U \subset \mathcal{T}_d$ be a monomial subregion, and let $T, T' \subset \mathcal{T}_d$ be any subregions such that $T \setminus U = T' \setminus U$. If $T \cap U$ and $T' \cap U$ are both tileable, then T is tileable if and only if T' is, by Lemma 3.1. In other words, replacing a tileable monomial subregion of a triangular region by a tileable monomial subregion of the same size does not affect tileability.

Using the above observation to reduce to the simplest case, we find a tileability criterion of triangular regions associated to monomial ideals.

Theorem 3.2. *Let $T = T_d(I)$ be a balanced triangular region, where $I \subset R$ is any monomial ideal. Then T is tileable if and only if T has no ∇ -heavy monomial subregions.*

Proof. Suppose T contains a ∇ -heavy monomial subregion U . That is, U has more downward-pointing triangles than upward-pointing triangles. Since the only triangles of $T \setminus U$ that share

an edge with U are downward-pointing triangles, it is impossible to cover every downward-pointing triangle of U with a lozenge. Thus, T is non-tileable.

Conversely, suppose T has no ∇ -heavy monomial subregions. In order to show that T is tileable, we may also assume that T has no non-trivial tileable monomial subregions by Lemma 3.1.

Consider any pair of touching or overlapping punctures in \mathcal{T}_d . The smallest monomial subregion U containing both punctures is tileable. (In fact, such a monomial region is uniquely tileable by lozenges.) If further triangles stemming from other punctures of T have been removed from U , then the resulting region $T \cap U$ becomes ∇ -heavy or empty. Thus, our assumptions imply that T has no overlapping and no touching punctures.

Now we proceed by induction on d . If $d \leq 2$, then T is empty or consists of one lozenge. Thus, it is tileable. Let $d \geq 3$, and let U be the monomial subregion of T associated to x , i.e., U consists of the upper $d - 1$ rows of T . Let L be the bottom row of T . If L does not contain part of a puncture of T , then L is \triangle -heavy forcing U to be a ∇ -heavy monomial subregion, contradicting an assumption on T . Hence, L must contain part of at least one puncture of T . See Figure 3.3(i).

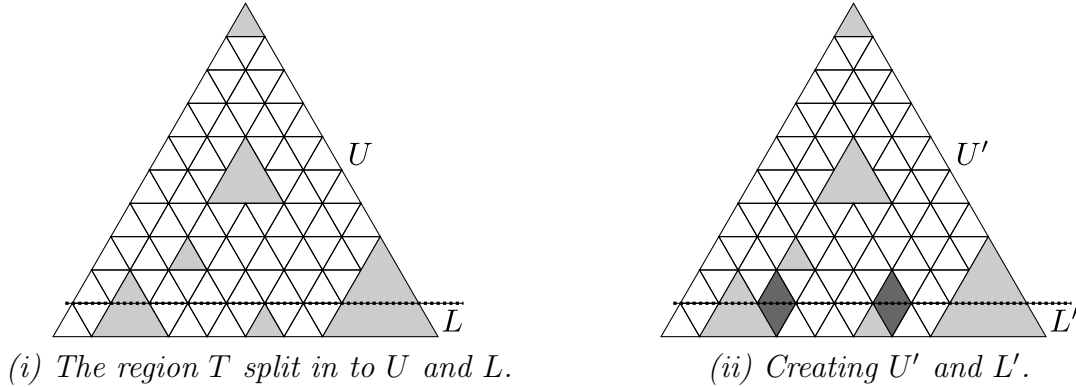


FIGURE 3.3. Illustrations for the proof of Theorem 3.2.

Place an up-down lozenge in T just to the right of each puncture along the bottom row *except* the farthest right puncture. Notice that putting in all these tiles is possible since punctures are non-overlapping and non-touching. Let $U' \subset U$ and $L' \subset L$ be the subregions that are obtained by removing the relevant upward-pointing and downward-pointing triangles of the added lozenges from U and L , respectively. See Figure 3.3(ii). Notice, L' is uniquely tileable.

As T and L' are balanced, so is U' . Assume U' contains a monomial subregion V' that is ∇ -heavy. Then $V' \neq U'$, and hence V' fits into a triangle of side length $d - 2$. Furthermore, the assumption on T implies that V' is not a monomial subregion of U . In particular, V' must be located at the bottom of U' . Let \tilde{V} be the smallest monomial subregion of U that contains V' . It is obtained from V' by adding suitable upward-pointing triangles that are parts of the added lozenges. Expand \tilde{V} down one row to a monomial subregion V of T . Thus, V fits into a triangle of side length $d - 1$ and is not ∇ -heavy. If V is balanced, then, by induction, V is tileable. However, we assumed T contains no such non-trivial regions. Hence, V is \triangle -heavy. Observe now that the region $V \cap L'$ is either balanced or has exactly one more upward-pointing triangle than downward-pointing triangles. Since V' is obtained

from V by removing $V \cap L$ and some of the added lozenges, it follows that V' cannot be ∇ -heavy, a contradiction.

Therefore, we have shown that each monomial subregion of U' is not ∇ -heavy. By induction on d , we conclude that U' is tileable. Using the lozenges already placed, along with the tiling of L' , we obtain a tiling of T . \square

Remark 3.3. The preceding proof yields a recursive construction of a canonical tiling of the triangular region. In fact, the tiling can be seen as minimal, in the sense of Subsection 5.2. Moreover, the theorem yields an exponential (in the number of punctures) algorithm to determine the tileability of a region.

Thurston [60] gave a linear (in the number of triangles) algorithm to determine the tileability of a *simply-connected region*, i.e., a region with a polygonal boundary. Thurston's algorithm also yields a minimal canonical tiling.

Let I be a monomial ideal of R whose punctures in \mathcal{T}_d (corresponding to the minimal generators of I having degree less than d) have side lengths that sum to m . Then we define the *over-puncturing coefficient* of I in degree d to be $\mathfrak{o}_d(I) = m - d$. If $\mathfrak{o}_d(I) < 0$, $\mathfrak{o}_d(I) = 0$, or $\mathfrak{o}_d(I) > 0$, then we call I *under-punctured*, *perfectly-punctured*, or *over-punctured* in degree d , respectively.

Let now $T = T_d(I)$ be a triangular region with punctures whose side lengths sum to m . Then we define similarly the *over-puncturing coefficient* of T to be $\mathfrak{o}_T = m - d$. If $\mathfrak{o}_T < 0$, $\mathfrak{o}_T = 0$, or $\mathfrak{o}_T > 0$, then we call T *under-punctured*, *perfectly-punctured*, or *over-punctured*, respectively. Note that $\mathfrak{o}_T = \mathfrak{o}_d(J(T)) \leq \mathfrak{o}_d(I)$, and equality is true if and only if the ideals I and $J(T)$ are the same in all degrees less than d .

Perfectly-punctured regions admit a numerical tileability criterion.

Corollary 3.4. *Let $T = T_d(I)$ be a triangular region. Then any two of the following conditions imply the third:*

- (i) T is perfectly-punctured;
- (ii) T has no over-punctured monomial subregions; and
- (iii) T is tileable.

Proof. If T is unbalanced, then T is not tileable. Moreover, if T is perfectly-punctured, then at least two punctures must overlap, thus creating an over-punctured monomial subregion. Hence, we may assume T is balanced for the remainder of the argument.

Suppose T is tileable. Then T has no ∇ -heavy monomial subregions, by Theorem 3.2. Thus every monomial subregion of T is not over-punctured if and only if no punctures of T overlap, i.e., T is perfectly-punctured.

If T is non-tileable, then T has a ∇ -heavy monomial subregion. Since every ∇ -heavy monomial subregion is also over-punctured, it follows that T has an over-punctured monomial subregion. \square

4. STABILITY OF SYZYGY BUNDLES

Let I be an Artinian ideal of $S = K[x_1, \dots, x_n]$ that is minimally generated by forms f_1, \dots, f_m . The *syzygy module* of I is the graded module $\text{syz } I$ that fits into the exact sequence

$$0 \rightarrow \text{syz } I \rightarrow \bigoplus_{i=1}^m S(-\deg f_i) \rightarrow I \rightarrow 0.$$

Its sheafification $\widetilde{\text{syz}}I$ is a vector bundle on \mathbb{P}^{n-1} , called the *syzygy bundle* of I . It has rank $m - 1$.

Semistability is an important property of a vector bundle. Let E be a vector bundle on projective space. The *slope* of E is defined as $\mu(E) := \frac{c_1(E)}{rk(E)}$. Furthermore, E is said to be *semistable* if the inequality $\mu(F) \leq \mu(E)$ holds for every coherent subsheaf $F \subset E$. If the inequality is always strict, then E is said to be *stable*.

Brenner established a beautiful characterisation of the semistability of syzygy bundles to monomial ideals. Since we only consider monomial ideals in this work, the following may be taken as the definition of (semi)stability herein.

Theorem 4.1. [7, Proposition 2.2 & Corollary 6.4] *Let I be an Artinian ideal in $K[x_1, \dots, x_n]$ that is minimally generated by monomials g_1, \dots, g_m , where K is a field of characteristic zero. Then I has a semistable syzygy bundle if and only if, for every proper subset J of $\{1, \dots, m\}$ with at least two elements, the inequality*

$$\frac{d_J - \sum_{j \in J} \deg g_j}{|J| - 1} \leq \frac{\sum_{i=1}^m \deg g_i}{m - 1}$$

holds, where d_J is the degree of the greatest common divisor of the g_j with $j \in J$. Further, I has a stable syzygy bundle if and only if the above inequality is always strict.

Notice that the right-hand side in the above inequalities is the slope of the syzygy bundle of I .

We use the above criterion to rephrase (semi)stability in the case of a monomial ideal in $K[x, y, z]$ in terms of the over-puncturing coefficients of ideals. To do this, we first reinterpret the slope. Throughout this section we continue to assume that K is a field of characteristic zero.

Lemma 4.2. *Let I be an Artinian ideal in $R = K[x, y, z]$ that is minimally generated by monomials g_1, \dots, g_m whose degrees are bounded above by d . Then*

$$\mu(\widetilde{\text{syz}}I) = -d + \frac{\mathfrak{o}_d(I)}{m - 1}.$$

Proof. Recall that the side length of the puncture associated to g_i in $T = T_d(I)$ is $d - \deg g_i$. Thus, $\mathfrak{o}_d(I) = \sum_{i=1}^m (d - \deg g_i) - d$. Hence we obtain

$$\begin{aligned} \mu(\widetilde{\text{syz}}I) &= \frac{-\sum_{i=1}^m \deg g_i}{m - 1} \\ &= \frac{[\sum_{i=1}^m (d - \deg g_i) - d] - d(m - 1)}{m - 1} \\ &= -d + \frac{\sum_{i=1}^m (d - \deg g_i) - d}{m - 1} \\ &= -d + \frac{\mathfrak{o}_d(I)}{m - 1}. \end{aligned}$$

□

Observe that $\mathfrak{o}_{d+1}(I) = \mathfrak{o}_d(I) + m - 1$.

Now we reinterpret Theorem 4.1 by using over-puncturing coefficients.

Corollary 4.3. *Let I be an Artinian ideal in $R = K[x, y, z]$ that is minimally generated by monomials g_1, \dots, g_m of degree at most d . For every proper subset J of $\{1, \dots, m\}$ with at least two elements, let I_J be the monomial ideal that is generated by $\{g_j/g \mid j \in J\}$, where $g = \gcd\{g_j \mid j \in J\}$.*

Then I has a semistable syzygy bundle if and only if, for every proper subset J of $\{1, \dots, m\}$ with at least two elements, the inequality

$$\frac{\mathfrak{o}_{d-\deg g}(I_J)}{|J| - 1} \leq \frac{\mathfrak{o}_d(I)}{m - 1}$$

holds. Furthermore, I has a stable syzygy bundle if and only if the above inequality is always strict.

Proof. Let J be a proper subset of $\{1, \dots, m\}$ with $n \geq 2$ elements. Then a computation similar to the one in Lemma 4.2 provides

$$\begin{aligned} \frac{\deg g - \sum_{j \in J} \deg g_j}{|J| - 1} &= \frac{-(n - 1) \deg g - \sum_{j \in J} (\deg g_j - \deg g)}{|J| - 1} \\ &= -\deg g + \frac{-\sum_{j \in J} \deg(g_j/g)}{|J| - 1} \\ &= -\deg g + (\deg g - d) + \frac{\mathfrak{o}_{T_{d-\deg g}}(I_J)}{|J| - 1} \\ &= -d + \frac{\mathfrak{o}_{d-\deg g}(I_J)}{|J| - 1}. \end{aligned}$$

Taking into account also Lemma 4.2, Theorem 4.1 shows that we need to compare $-d + \frac{\mathfrak{o}_{d-\deg g}(I_J)}{|J| - 1}$ and $-d + \frac{\mathfrak{o}_d(I)}{m - 1}$. \square

In order to better interpret the last result we slightly extend the concept of a triangular region $T_d(I)$ in the remainder of this section. Label the vertices in \mathcal{T}_d by monomials of degree d such that the label of each unit triangle is the greatest common divisor of its vertex labels. Then a minimal monomial generator of I with degree d corresponds to a vertex of \mathcal{T}_d that is removed in $T_d(I)$. We consider this removed vertex as a puncture of side length zero. Observe that this is in line with our general definition of the side length of a puncture (see Subsection 2.2). Now Corollary 4.3 can be rephrased as saying that the syzygy bundle of I is semistable if and only if the “average” over-puncturing per puncture for any non-trivial collection of punctures of I (restricted to their smallest containing triangle) is at most the “average” over-puncturing per puncture for the entire ideal I .

Example 4.4. We illustrate this point of view by giving quick proofs of some known results.

(i) ([1, Theorem 0.2]) *For each $d \geq 1$, the syzygy bundle of $(x, y, z)^d$ is stable.*

Proof. Consider $T_d = T_d((x, y, z)^{d-1})$, that is, T_d is obtained from \mathcal{T}_d by removing all $\binom{d+1}{2}$ upward-pointing triangles. Then

$$\frac{\mathfrak{o}_{T_d}}{\binom{d+1}{2} - 1} = \frac{\binom{d+1}{2} - d}{\binom{d+1}{2} - 1} = \frac{d}{d + 2}.$$

Now we consider the average over-puncturing of any non-trivial collection of punctures in T in a triangle of side length e , where $2 \leq e \leq d$:

- (a) If $e < d$, then the average over-puncturing is maximised when all the punctures in the triangle are present, i.e., when the associated monomial subregion is T_e .

Clearly, $\frac{e}{e+2} < \frac{d}{d+2}$ if $2 \leq e < d$.

- (b) If $e = d$, then it is maximised when all but one puncture is present. The over-puncturing is thus

$$\frac{\binom{d+1}{2} - d - 1}{\binom{d+1}{2} - 2} < \frac{\binom{d+1}{2} - d}{\binom{d+1}{2} - 1}. \quad \square$$

- (ii) ([7, Corollary 7.1] & [28, Lemma 2.1]) *Let $I = (x^c, y^{c-a}, z^{c-b})$ be a monomial complete intersection where, without loss of generality, $0 \leq a \leq b < c$. Then $\widetilde{\text{syz}}I$ is semistable (or stable) if and only if the punctures in $T_c(I)$ do not overlap (or touch).*

Proof. Notice that $T_c(I)$ has a puncture of side length zero at its top, corresponding to x^c . Thus, the average over-puncturing of $T_c(I)$ is $\frac{1}{2}(a + b - c)$ and the average over-puncturing of the three non-trivial collections of punctures is $a - c$, $b - c$, and $a + b - c$ for $T_c(x^c, y^{c-a})$, $T_c(x^c, z^{c-b})$, and $T_c(y^{c-a}, z^{c-b})$, respectively. The latter is maximised at $a + b - c$.

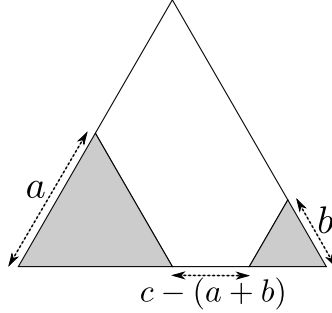


FIGURE 4.1. The region $T_c(x^c, y^{c-a}, z^{c-b})$, where $0 \leq a \leq b < c$.

Using Corollary 4.3, we see that I has a semistable (stable) syzygy bundle if and only if $\frac{1}{2}(a + b - c) \geq a + b - c$ (strictly), i.e., $c \geq a + b$ (strictly). Interpreting this in $T_c(I)$ (see Figure 4.1) yields the desired conclusion. \square

Using Corollary 3.4, we see that semistability is strongly related to tileability of a region.

Theorem 4.5. *Let I be an Artinian ideal in $R = K[x, y, z]$ generated by monomials whose degrees are bounded above by d , and let $T = T_d(I)$. If T is non-empty, then any two of the following conditions imply the third:*

- (i) I is perfectly-punctured;
- (ii) T is tileable; and
- (iii) $\widetilde{\text{syz}}I$ is semistable.

Proof. Assume T is perfectly-punctured, that is, $\mathfrak{o}_d(T) = 0$. By Corollary 3.4, T is tileable if and only if T has no over-punctured monomial subregions. The latter condition is equivalent to every monomial subregion of T having a non-positive over-puncturing coefficient. By Corollary 4.3, this is equivalent to $\widetilde{\text{syz}}I$ being semistable.

Now assume I is not perfectly-punctured, but T is tileable. We have to show that $\widetilde{\text{syz}}I$ is not semistable.

Observe that \mathcal{T}_d has exactly d more upward-pointing triangles than downward-pointing triangles. It follows that every balanced monomial subregion of \mathcal{T}_d cannot be under-punctured. Since T is balanced, but not perfectly punctured we conclude that T is over-punctured. Using again that T is balanced, T must have overlapping punctures. Consider two such overlapping punctures of T . Then the smallest monomial subregion U containing these two punctures does not overlap with any other puncture of T with positive side length (because it is not ∇ -heavy by Theorem 3.2) and is uniquely tileable. Hence $T' = T \setminus U$ is tileable (see Lemma 3.1) and $0 \leq \mathfrak{o}_{T'} < \mathfrak{o}_d(I)$. If T' is still over-punctured, then we repeat the above replacement procedure until we get a perfectly-punctured monomial subregion of T . Abusing notation slightly, denote this region by T' . Let J be the largest monomial ideal containing I and with generators whose degrees are bounded above by d such that $T' = T_d(J)$. Observe that $\mathfrak{o}_d(J) = \mathfrak{o}_{T'} = 0$.

Notice that a single replacement step above amounts to replacing the triangular region to an ideal I' by the region to the ideal (I', f) , where f is the greatest common divisor of a family of minimal generators of I' having degree less than d .

Assume now that T' is empty. By the above considerations, this means that I has a family of minimal generators, say, g_1, \dots, g_t of degrees $d - a_1, \dots, d - a_t < d$ that are relatively prime and whose corresponding punctures form two overlapping punctures of T . Thus, $a_1 + \dots + a_t > d$ (see Example 4.4(ii)). Furthermore, all other minimal generators of I , of which there must be at least one as I is Artinian, must have degree d since T is balanced. Hence the average over-puncturing of I is

$$\frac{\mathfrak{o}_d(I)}{m-1} = \frac{a_1 + \dots + a_t - d}{m-1} \leq \frac{a_1 + \dots + a_t - d}{t},$$

where $m \geq t+1$ is the number of minimal generators of I . However, the average over-puncturing corresponding to the ideal I' generated by g_1, \dots, g_t is

$$\frac{\mathfrak{o}_d(I')}{t-1} = \frac{a_1 + \dots + a_t - d}{t-1} > \frac{a_1 + \dots + a_t - d}{t}.$$

Hence, Corollary 4.3 shows that $\widetilde{\text{syz}}I$ is not semistable.

It remains to consider the case where T' is not empty, i.e., J is a proper ideal of R . Let g_1, \dots, g_m and f_1, \dots, f_n be the minimal monomial generators of I and J , respectively. Partition the generating set of I into $F_j = \{g_i \mid g_i \text{ divides } f_j\}$. Notice $f_j = \gcd\{F_j\}$. In particular, $n > 1$ as I is an Artinian ideal.

Set $\mathfrak{o}_j = \sum_{g \in F_j} (d - \deg g) - (d - \deg f_j)$. Observe $\mathfrak{o}_j \geq 0$ as the region associated to the ideal generated by F_j is tileable, hence not under-punctured. Moreover,

$$\begin{aligned} \mathfrak{o}_d(J) &= \sum_{j=1}^n (d - \deg f_j) - d = \sum_{j=1}^n \left(\sum_{g \in F_j} (d - \deg g) - \mathfrak{o}_j \right) - d \\ &= \sum_{j=1}^n \sum_{g \in F_j} (d - \deg g) - d - \sum_{j=1}^n \mathfrak{o}_j. \end{aligned}$$

As $\mathfrak{o}_d(J) = 0$, we conclude that $\mathfrak{o}_d(I) = \sum_{j=1}^n \mathfrak{o}_j$ and, in particular, $\mathfrak{o}_d(I) \geq \mathfrak{o}_j$ for each j .

Assume $m \cdot \mathbf{o}_j < \#F_j \cdot \mathbf{o}_d(I)$ for all j . Then $m \sum_{j=1}^n \mathbf{o}_j < \mathbf{o}_d(I) \sum_{j=1}^n \#F_j$, but this implies $m \cdot \mathbf{o}_d(I) < m \cdot \mathbf{o}_d(I)$, which is absurd. Hence, there is some k such that $m \cdot \mathbf{o}_k \geq \#F_k \cdot \mathbf{o}_d(I)$. Since $\mathbf{o}_d(I) \geq \mathbf{o}_k$ it follows that $\frac{\mathbf{o}_k}{\#F_k - 1} > \frac{\mathbf{o}_T}{m - 1}$. Indeed, this is immediate if $\mathbf{o}_d(I) > \mathbf{o}_k$. If $\mathbf{o}_d(I) = \mathbf{o}_k$, then it is also true because $\#F_k < m$. Now Corollary 4.3 provides that $\widetilde{\text{syz}}I$ is not semistable. \square

We get the following criterion when focusing solely on the triangular region. Recall that $J(T)$ denotes the monomial ideal of a triangular region T (see Subsection 2.5).

Corollary 4.6. *Let I be an Artinian ideal in $R = K[x, y, z]$ generated by monomials whose degrees are bounded above by d , and let $T = T_d(I)$. Assume T is non-empty and tileable.*

- (i) *If $I \neq I + J(T)$, then $\widetilde{\text{syz}}I$ is not semistable.*
- (ii) *$\widetilde{\text{syz}}(I + J(T))$ is semistable if and only if T is perfectly-punctured.*

Proof. Since T is balanced, we get $0 \leq \mathbf{o}_T = \mathbf{o}_d(J(T)) = \mathbf{o}_d(I + J(T))$. Hence Theorem 4.5 provides our assertions. Note for claim (i) that $I \neq I + J(T)$ implies $\mathbf{o}_d(I + J(T)) < \mathbf{o}_d(I)$. \square

For stability, we obtain the following result.

Proposition 4.7. *Let I be an Artinian ideal in $R = K[x, y, z]$ generated by monomials whose degrees are bounded above by d . If $T = T_d(I)$ is non-empty, tileable, and perfectly-punctured, then $\widetilde{\text{syz}}(I + J(T))$ is stable if and only if every proper monomial subregion of T is under-punctured.*

Proof. We may assume $I = I + J(T)$. As T is perfectly-punctured, we have that $\mathbf{o}_d(I) = \mathbf{o}_T = 0$. Using Corollary 4.3, we see that $\widetilde{\text{syz}}I$ is stable if and only if $\mathbf{o}_{T_d - \deg g(I_J)} < 0$, where $g = \gcd\{J\}$, for all proper subsets J of the set of minimal generators of I . This is equivalent to every proper monomial subregion of T being under-punctured. \square

By the preceding theorem and proposition, we have an understanding of semistability and stability for perfectly-punctured triangular regions. However, when a region is over-punctured and non-tileable more information is needed to decide semistability.

Example 4.8. There are monomial ideals with stable syzygy bundles whose corresponding triangular regions are over-punctured and non-tileable. See Example 4.4(i) and Figure 4.2(i) for a specific example.

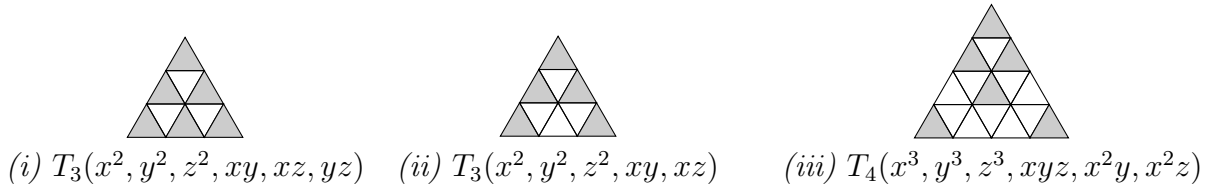


FIGURE 4.2. Over-punctured, non-tileable regions and various levels of stability.

Moreover, the ideal (x^2, y^2, z^2, xy, xz) has a semistable, but non-stable syzygy bundle (the monomial subregion associated to x breaks stability), and the ideal $(x^3, y^3, z^3, xyz, x^2y, x^2z)$ has a non-semistable syzygy bundle (the monomial subregion associated to x^2 breaks semistability). Both of their triangular regions, see Figures 4.2(ii) and (iii), respectively, are over-punctured and non-tileable.

5. SIGNED LOZENGE TILINGS AND ENUMERATIONS

In Section 3 we considered whether a triangular region $T_d(I)$ is tileable by lozenges. Now we want to *enumerate* the tilings of tileable regions $T_d(I)$. In fact, we introduce two ways for assigning a sign to a lozenge tiling and then compare the resulting enumerations.

In order to derive the (unsigned) enumeration, we consider the enumeration of perfect matchings of an associated bipartite graph. If we consider the bi-adjacency matrix, a zero-one matrix, of the bipartite graph, then the permanent of the matrix yields the desired enumeration. However, the determinant of this matrix yields a (possibly different) integer, which may be negative. We consider this a *signed* enumeration of the perfect matchings of the graph, and hence of lozenge tilings.

We derive a different *signed* enumeration of the lozenge tilings by considering the enumeration of families of non-intersecting lattice paths on an associated finite sub-lattice of \mathbb{Z}^2 . Using the Lindström-Gessel-Viennot Theorem ([39], [23]; see Theorem 5.7), we generate a binomial matrix for the finite sub-lattice with a determinant that gives a signed enumeration of families of non-intersecting lattice paths, hence of lozenge tilings. The two signed enumerations appear to be different, but we show that they are indeed the same, up to sign.

5.1. Perfect matchings.

A subregion $T(G) \subset \mathcal{T}_d$ can be associated to a bipartite planar graph G that is an induced subgraph of the honeycomb graph. Lozenge tilings of $T(G)$ can be then associated to perfect matchings on G . The connection was used by Kuperberg in [35], the earliest citation known to the authors, to study symmetries on plane partitions. Note that $T(G)$ is often called the *dual graph* of G in the literature (e.g., [11], [12], and [20]). Here we begin with a subregion T and then construct a graph G .

Let $T \subset \mathcal{T}_d$ be any subregion. As above, we consider T as a union of unit triangles. We associate to T a bipartite graph. First, place a vertex at the center of each triangle. Let B be the set of centers of the downward-pointing triangles, and let W be the set of centers of the upward-pointing triangles. Consider both sets ordered by the reverse-lexicographic ordering applied to the monomial labels of the corresponding triangles (see Subsection 2.2). The *bipartite graph associated to T* is the bipartite graph $G(T)$ on the vertex set $B \cup W$ that has an edge between vertices $B_i \in B$ and $W_j \in W$ if the corresponding upward- and downward-pointing triangle share an edge. In other words, edges of $G(T)$ connect vertices of adjacent triangles. See Figure 5.1(i).

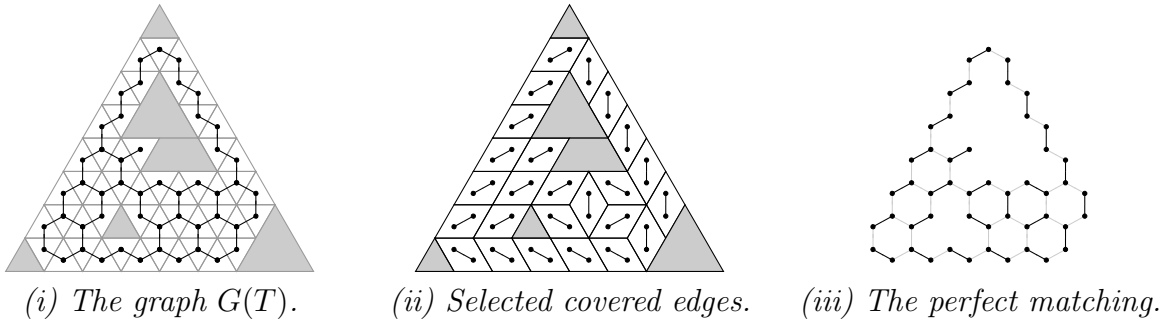


FIGURE 5.1. Given the tiling τ in Figure 3.1 of T , we construct the perfect matching π of the bipartite graph $G(T)$ associated to τ .

Using the above ordering of the vertices, we define the *bi-adjacency matrix* of T as the bi-adjacency matrix $Z(T) := Z(G(T))$ of the graph $G(T)$. It is the zero-one matrix $Z(T)$ of size $\#B \times \#W$ with entries $Z(T)_{(i,j)}$ defined by

$$Z(T)_{(i,j)} = \begin{cases} 1 & \text{if } (B_i, W_j) \text{ is an edge of } G(T) \\ 0 & \text{otherwise.} \end{cases}$$

Remark 5.1. Note that that $Z(T)$ is a square matrix if and only if the region T is balanced. Observe also that the construction of $G(T)$ and $Z(T)$ do not require any restrictions on T . In particular, T need not be balanced and so $Z(T)$ need not be square. This generality is needed in Section 8.2.

A *perfect matching* of a graph G is a set of pairwise non-adjacent edges of G such that each vertex is matched. There is well-known bijection between lozenge tilings of a balanced subregion T and perfect matchings of $G(T)$. A lozenge tiling τ is transformed in to a perfect matching π by overlaying the triangular region T on the bipartite graph $G(T)$ and selecting the edges of the graph that the lozenges of τ cover. See Figures 5.1(ii) and (iii) for the overlayed image and the perfect matching by itself, respectively.

Remark 5.2. The graph $G(T)$ is a “honeycomb graph,” a type of graph that has been studied, especially for its perfect matchings.

- (i) In particular, honeycomb graphs are investigated for their connections to physics. Honeycomb graphs model the bonds in dimers (polymers with only two structural units), and perfect matchings correspond to so-called *dimer coverings*. Kenyon [31] gave a modern recount of explorations on dimer models, including random dimer coverings and their limiting shapes. The recent memoir [12] of Ciucu, which has many connections to this paper (see Section 7), describes further results in this direction.
- (ii) Kasteleyn [30] provided, in 1967, a general method for computing the number of perfect matchings of a planar graph by means of a determinant. In the following proposition, we compute the number of perfect matchings on $G(T)$ by means of a permanent.

Recall that the *permanent* of an $n \times n$ matrix $M = (M_{(i,j)})$ is given by

$$\text{perm } M := \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n M_{(i, \sigma(i))}.$$

Proposition 5.3. *Let $T \subset \mathcal{T}_d$ be a non-empty balanced subregion. Then the lozenge tilings of T and the perfect matchings of $G(T)$ are both enumerated by $\text{perm } Z(T)$.*

Proof. As T is balanced, $Z(T)$ is a square zero-one matrix. Each non-zero summand of $\text{perm } Z(T)$ corresponds to a perfect matching, as it corresponds to a bijection between the two colour classes B and W of $G(T)$ (determined by the downward- and upward-pointing triangles of T). Hence, $\text{perm } Z(T)$ enumerates the perfect matchings of $G(T)$, and thus the tilings of T . \square

Recall that the *determinant* of an $n \times n$ matrix M is given by

$$\det M := \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n \operatorname{sgn} \sigma M_{(i, \sigma(i))}.$$

Each non-zero summand of the determinant of M is given a sign based on the signature (or sign) of the permutation associated to it. We take the convention that the permanent and determinant of a 0×0 matrix is one.

By the proof of Proposition 5.3, each lozenge tiling τ corresponds to a perfect matching π of $G(T)$, that is, a bijection $\pi : B \rightarrow W$. Considering π as a permutation on $\#\Delta(T) = \#\nabla(T)$ letters, it is natural to assign a sign to each lozenge tiling using the signature of the permutation π .

Definition 5.4. Let $T \subset \mathcal{T}_d$ be a non-empty balanced subregion. Then we define the *perfect matching sign* of a lozenge tiling τ of T as $\operatorname{msgn} \tau := \operatorname{sgn} \pi$, where $\pi \in \mathfrak{S}_{\#\Delta(T)}$ is the perfect matching determined by τ .

It follows that the determinant of $Z(T)$ gives an enumeration of the *perfect matching signed lozenge tilings* of T .

Theorem 5.5. Let $T \subset \mathcal{T}_d$ be a non-empty balanced subregion. Then the perfect matching signed lozenge tilings of T are enumerated by $\det Z(T)$, that is,

$$\sum_{\tau \text{ tiling of } T} \operatorname{msgn} \tau = \det Z(T).$$

Example 5.6. Consider the triangular region $T = T_6(x^3, y^4, z^5)$, as seen in the first picture of Figure 5.3. Then $Z(T)$ is the 11×11 matrix

$$Z(T) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

We note that $\operatorname{perm} Z(T) = \det Z(T) = 10$. Thus, T has exactly 10 lozenge tilings, all of which have the same sign. We derive a theoretical explanation for this fact in the following two subsections.

5.2. Families of non-intersecting lattice paths.

We follow [13, Section 5] (similarly, [21, Section 2]) in order to associate to a subregion $T \subset \mathcal{T}_d$ a finite set $L(T)$ that can be identified with a subset of the lattice \mathbb{Z}^2 . Abusing notation, we refer to $L(T)$ as a sub-lattice of \mathbb{Z}^2 . We then translate lozenge tilings of T into families of non-intersecting lattice paths on $L(T)$.

We first construct $L(T)$ from T . Place a vertex at the midpoint of the edge of each triangle of T that is parallel to the upper-left boundary of the triangle \mathcal{T}_d . These vertices form $L(T)$. We will consider paths in $L(T)$. There we think of rightward motion parallel to the bottom edge of \mathcal{T}_d as “horizontal” and downward motion parallel to the upper-right edge of \mathcal{T}_d as “vertical” motion. If one simply orthogonalises $L(T)$ with respect to the described “horizontal” and “vertical” motions, then we can consider $L(T)$ as a finite sub-lattice of \mathbb{Z}^2 . As we can translate $L(T)$ in \mathbb{Z}^2 and not change its properties, we may assume that the vertex associated to the lower-left triangle of \mathcal{T}_d is the origin. Notice that each vertex of $L(T)$ is on the upper-left edge of an upward-pointing triangle of \mathcal{T}_d (even if this triangle is not present in T). We use the monomial label of this upward-pointing triangle to specify a vertex of $L(T)$. Under this identification the mentioned orthogonalisation of $L(T)$ moves the vertex associated to the monomial $x^a y^b z^{d-1-(a+b)}$ in $L(T)$ to the point $(d-1-b, a)$ in \mathbb{Z}^2 .

We next single out special vertices of $L(T)$. We label the vertices of $L(T)$ that are only on upward-pointing triangles in T , from smallest to largest in the reverse-lexicographic order, as A_1, \dots, A_m . Similarly, we label the vertices of $L(T)$ that are only on downward-pointing triangles in T , again from smallest to largest in the reverse-lexicographic order, as E_1, \dots, E_n . See Figure 5.2(i). We note that there are an equal number of vertices A_1, \dots, A_m and E_1, \dots, E_n if and only if the region T is balanced. This follows from the fact these vertices are precisely the vertices of $L(T)$ that are in exactly one unit triangle of T .

A *lattice path* in a lattice $L \subset \mathbb{Z}^2$ is a finite sequence of vertices of L so that all single steps move either to the right or down. Given any vertices $A, E \in \mathbb{Z}^2$, the number of lattice paths in \mathbb{Z}^2 from A to E is a binomial coefficient. In fact, if A and E have coordinates $(u, v), (x, y) \in \mathbb{Z}^2$ as above, there are $\binom{x-u+v-y}{x-u}$ lattice paths from A to E as each path has $x-u+v-y$ steps and $x-u \geq 0$ of these must be horizontal steps.

Using the above identification of $L(T)$ as a sub-lattice of \mathbb{Z}^2 , a *lattice path* in $L(T)$ is a finite sequence of vertices of $L(T)$ so that all single steps move either to the East or to the Southeast. The *lattice path matrix* of T is the $m \times n$ matrix $N(T)$ with entries $N(T)_{(i,j)}$ defined by

$$N(T)_{(i,j)} = \# \text{lattice paths in } \mathbb{Z}^2 \text{ from } A_i \text{ to } E_j.$$

Thus, the entries of $N(T)$ are binomial coefficients.

Next we consider several lattice paths simultaneously. A *family of non-intersecting lattice paths* is a finite collection of lattice paths such that no two lattice paths have any points in common. We call a family of non-intersecting lattice paths *minimal* if every path takes vertical steps before it takes horizontal steps, whenever possible. That is, every time a horizontal step is followed by a vertical step, then replacing these with a vertical step followed by a horizontal step would cause paths in the family to intersect.

Assume now that the subregion T is balanced, so $m = n$. Let Λ be a family of m non-intersecting lattice paths in $L(T)$ from A_1, \dots, A_m to E_1, \dots, E_m . Then Λ determines a permutation $\lambda \in \mathfrak{S}_m$ such that the path in Λ that begins at A_i ends at $E_{\lambda(i)}$.

Now we are ready to apply a beautiful theorem relating enumerations of signed families of non-intersecting lattice paths and determinants. In particular, we use a theorem first given by Lindström in [39, Lemma 1] and stated independently in [23, Theorem 1] by Gessel and Viennot. Stanley gives a very nice exposition of the topic in [58, Section 2.7].

Theorem 5.7. [39, Lemma 1] & [23, Theorem 1] *Assume $T \subset \mathcal{T}_d$ is a non-empty balanced subregion with identified lattice points $A_1, \dots, A_m, E_1, \dots, E_m \in L(T)$ as above. Then*

$$\det N(T) = \sum_{\lambda \in \mathfrak{S}_m} \operatorname{sgn}(\lambda) P_{\lambda}^+(A \rightarrow E),$$

where, for each permutation $\lambda \in \mathfrak{S}_m$, $P_{\lambda}^+(A \rightarrow E)$ is the number of families of non-intersecting lattice paths with paths in $L(T)$ going from A_i to $E_{\lambda(i)}$.

We now use a well-know bijection between lozenge tilings of T and families of non-intersecting lattice paths from A_1, \dots, A_m to E_1, \dots, E_m ; see, e.g., the survey [53]. Let τ be a lozenge tiling of T . Using the lozenges of τ as a guide, we connect each pair of vertices of $L(T)$ that occur on a single lozenge. This generates the family of non-intersecting lattice paths Λ of $L(T)$ corresponding to τ . See Figures 5.2(ii) and (iii) for the overlaid image and the family of non-intersecting lattice paths by itself, respectively.

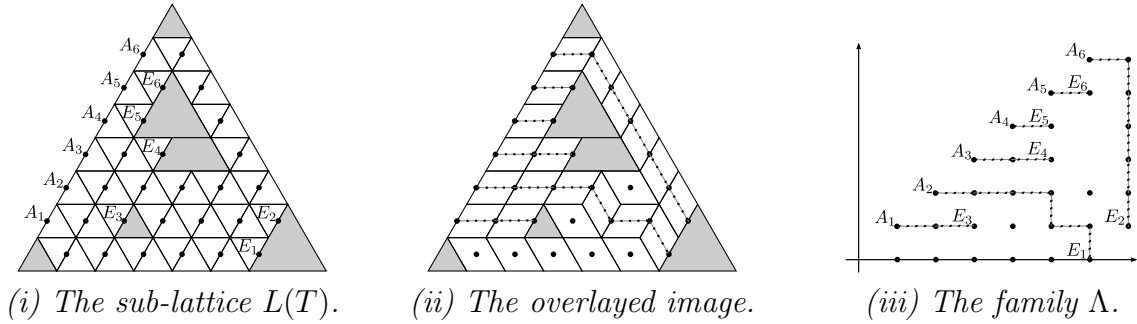


FIGURE 5.2. The family of non-intersecting lattice paths Λ associated to the tiling τ in Figure 3.1.

This bijection provides another way for assigning a sign to a lozenge tiling, this time using the signature of the permutation λ .

Definition 5.8. Let $T \subset \mathcal{T}_d$ be a non-empty balanced subregion as above, and let τ be a lozenge tiling of T . Then we define the *lattice path sign* of τ as $\operatorname{lpsgn} \tau := \operatorname{sgn} \lambda$, where $\lambda \in \mathfrak{S}_m$ is the permutation such that, for each i , the lattice path determined by τ that starts at A_i ends at $E_{\lambda(i)}$.

It follows that the determinant of $N(T)$ gives an enumeration of the *lattice path signed lozenge tilings* of T .

Theorem 5.9. Let $T \subset \mathcal{T}_d$ be a non-empty balanced subregion. Then the lattice path signed lozenge tilings of T are enumerated by $\det N(T)$, that is,

$$\sum_{\tau \text{ tiling of } T} \operatorname{lpsgn} \tau = \det N(T).$$

Remark 5.10. Notice that we can use the above construction to assign, for each subregion T , three (non-trivially) different lattice path matrices. The matrix $N(T)$ from Theorem 5.9 is one of these matrices, and the other two are the $N(\cdot)$ matrices of the 120° and 240° rotations of T . See Figure 5.3 for an example.

Finally, we note that in [53], Propp gave a history of the connections between lozenge tilings (of non-punctured hexagons), perfect matchings, plane partitions, and non-intersecting lattice paths.

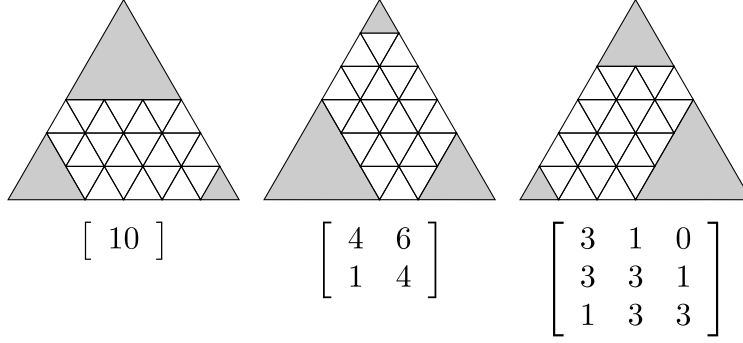


FIGURE 5.3. The triangular region $T_6(x^3, y^4, z^5)$ and its rotations, along with the $N(\cdot)$.

5.3. Interlude of signs.

We now have two different signs, the perfect matching sign and the lattice path sign, associated to each lozenge tiling of a balanced region T . In the case where T is a triangular region, we demonstrate in this subsection that the signs are equivalent, up to a scaling factor dependent only on $T = T_d(I)$. In particular, the main result of this section (Theorem 5.17) states that $|\det Z(T)| = |\det N(T)|$. In order to prove this theorem, we first make a few definitions. Throughout this subsection $T = T_d(I)$ is a tileable triangular region as introduced in Section 2. In particular, T is balanced, and each puncture of T has positive side length.

5.3.1. Resolution of punctures.

The first is a tool to remove a puncture from a triangular region, relative to some tiling, in a controlled fashion.

First, suppose that $T \subset \mathcal{T}_d$ has at least one puncture, call it \mathcal{P} , that is not overlapped by any other puncture of T . Let τ be some lozenge tiling of T , and denote by k the side length of \mathcal{P} . Informally, we will replace T by a triangular region in \mathcal{T}_{d+2k} , where the place of the puncture \mathcal{P} of T is taken by a tiled regular hexagon of side length k and three corridors to the outer vertices of \mathcal{T}_{d+2k} that are all part of the new region.

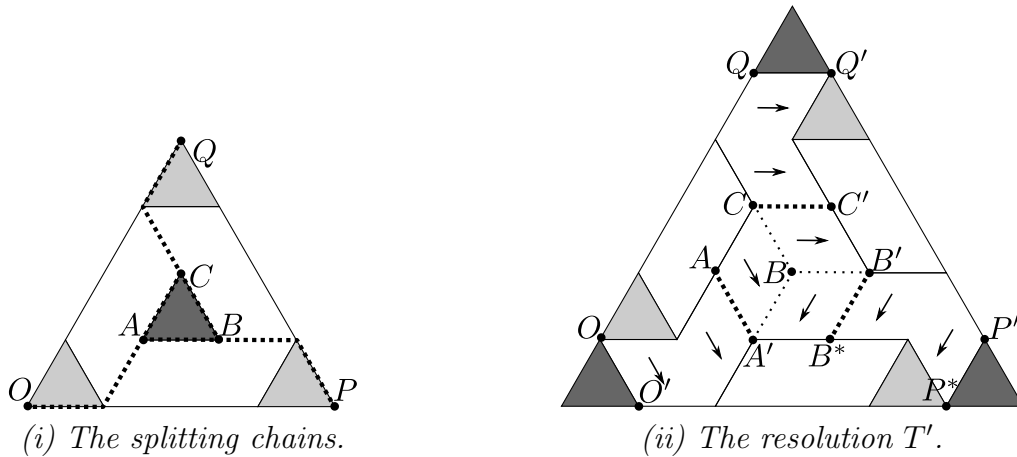


FIGURE 5.4. The abstract resolution of a puncture.

As above, we label the vertices of \mathcal{T}_d such that the label of each unit triangle is the greatest common divisor of its vertex labels. For ease of reference, we denote the lower-left, lower-right, and top vertex of the puncture \mathcal{P} by A, B , and C , respectively. Similarly, we denote the lower-left, lower-right, and top vertex of \mathcal{T}_d by O, P , and Q , respectively. Now we select three chains of unit edges such that each edge is either in T or on the boundary of a puncture of T . We start by choosing chains connecting A to O , B to P , and C to Q , respectively, subject to the following conditions:

- The chains do not cross, that is, do not share any vertices.
- There are no redundant edges, that is, omitting any unit edge destroys the connection between the desired end points of a chain.
- There are no moves to the East or Northeast on the lower-left chain OA .
- There are no moves to the West or Northwest on the lower-right chain PB .
- There are no moves to the Southeast or Southwest on the top chain CQ .

For these directions we envision a particle that starts at a vertex of the puncture and moves on a chain to the corresponding corner vertex of \mathcal{T}_d .

Now we connect the chains OA and CQ to a chain of unit edges $OACQ$ by using the Northeast edge of \mathcal{P} . Similarly we connect the chains OA and BP to a chain $OABP$ by using the horizontal edge of \mathcal{P} , and we connect PB and CQ to the chain $PBCQ$ by using the Northwest side of \mathcal{P} . These three chains subdivide \mathcal{T}_d into four regions. Part of the boundary of three of these regions is an edge of \mathcal{T}_d . The fourth region, the central one, is the area of the puncture \mathcal{P} . See Figure 5.4(i) for an illustration.

Now consider $T \subset \mathcal{T}_d$ as embedded into \mathcal{T}_{d+2k} such that the original region \mathcal{T}_d is identified with the triangular region $T_{d+2k}(x^k y^k)$. Retain the names A, B, C, O, P , and Q for the specified vertices of T as above. We create new chains of unit edges in \mathcal{T}_{d+2k} .

First, multiply each vertex in the chain $PBCQ$ by $\frac{z^k}{y^k}$ and connect the resulting vertices to a chain $P'B'C'Q'$ that is parallel to the chain $PBCQ$. Here P', B', C' , and Q' are the images of P, B, C , and Q under the multiplication by $\frac{z^k}{y^k}$. Informally, the chain $P'B'C'Q'$ is obtained by moving the chain $PBCQ$ just k units to the East.

Second, multiply each vertex in the chain OA by $\frac{z^k}{x^k}$ and connect the resulting vertices to a chain $O'A'$ that is parallel to the chain OA . Here A' and O' are the points corresponding to A and O . Informally the chain $O'A'$ is obtained by moving the chain OA just k units to the Southeast.

Third, multiply each vertex in the chain $P'B'$ by $\frac{y^k}{x^k}$ and connect the resulting vertices to a chain P^*B^* that is parallel to the chain $P'B'$, where P^* and B^* are the images of P' and B' , respectively. Thus, P^*B^* is k units to the Southwest of the chain $P'B'$. Connecting A' and B^* by horizontal edges, we obtain a chain $O'A'B^*P^*$ that has the same shape as the chain $OABP$. See Figure 5.4(ii) for an illustration.

We are ready to describe the desired triangular region $T' \subset \mathcal{T}_{d+2k}$ along with a tiling. Place lozenges and punctures in the region bounded by the chain $OACQ$ and the Northeast boundary of \mathcal{T}_{d+2k} as in the corresponding region of T . Similarly place lozenges and punctures in the region bounded by the chain $P'B'C'Q'$ and the Northwest boundary of \mathcal{T}_{d+2k} as in the corresponding region of T that is bounded by $PBCQ$. Next, place lozenges and punctures in the region bounded by the chain $O'A'B^*P^*$ and the horizontal boundary of \mathcal{T}_{d+2k} as in the exterior region of T that is bounded by $OABP$. Observe that corresponding vertices of the parallel chains BCQ and $B'C'Q'$ can be connected by horizontal edges. The region

between two such edges that are one unit apart is uniquely tileable. This gives a lozenge tiling for the region between the two chains. Similarly, the corresponding vertices of the parallel chains OAC and $O'A'C'$ can be connected by Southeast edges. Respecting these edges gives a unique lozenge tiling for the region between the chains OAC and $O'A'C'$. In a similar fashion, the corresponding vertices of the parallel chains $P'B'$ and P^*B^* can be connected by Southwest edges, which we use as a guide for a lozenge tiling of the region between the two chains. Finally, the rhombus with vertices A', B^*, B' , and B admits a unique lozenge tiling. Let τ' the union of all the lozenges we placed in \mathcal{T}_{d+2k} , and denote by T' the triangular region that is tiled by τ' . Thus, $T' \subset \mathcal{T}_{d+2k}$ has a puncture of side length k at each corner of \mathcal{T}_{d+2k} . See Figure 5.5 for an illustration of this. We call the region T' with its tiling τ' a *resolution of the puncture \mathcal{P} in T relative to τ* or, simply, a *resolution of \mathcal{P}* .

Observe that the tiles in τ' that were not carried over from the tiling τ are in the region that is the union of the regular hexagon with vertices A, A', B^*, B', C' and C and the regions between the parallel chains OA and $O'A'$, CQ and $C'Q'$ as well as $P'B'$ and P^*B^* . We refer to the latter three regions as the *corridors* of the resolution. Furthermore, we call the chosen chains OA , PB , and CQ the *splitting chains* of the resolution. The resolution blows up each splitting chain to a corridor of width k .

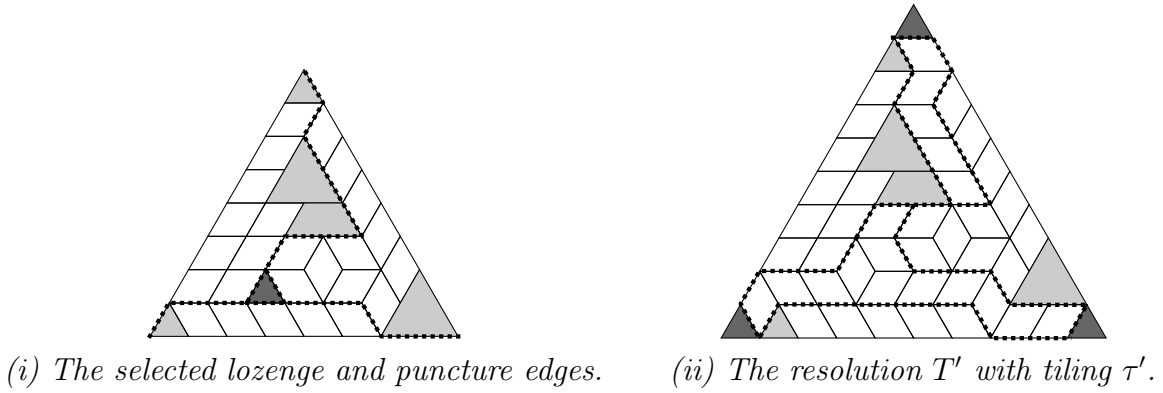


FIGURE 5.5. A resolution of the puncture associated to xy^4z^2 , given the tiling τ in Figure 3.1 of T .

Second, suppose the puncture \mathcal{P} in T is overlapped by another puncture of T . Then we cannot resolve \mathcal{P} using the above technique directly as it would result in a non-triangular region. Thus, we adapt it. Since T is balanced, \mathcal{P} is overlapped by exactly one puncture of T (see Theorem 3.2). Let U be the smallest monomial subregion of T that contains both punctures. We call U the *minimal covering region* of the two punctures. It is uniquely tileable, and we resolve the puncture U of $T \setminus U$. Notice that the lozenges inside U are lost during resolution. However, since U is uniquely tileable, they are recoverable from the two punctures of T in U . See Figure 5.6 for an illustration.

5.3.2. Cycles of lozenges.

Let τ be some tiling of T . An n -cycle (of lozenges) σ in τ is an ordered collection of distinct lozenges ℓ_1, \dots, ℓ_n of τ such that the downward-pointing triangle of ℓ_i is adjacent to the upward-pointing triangle of ℓ_{i+1} for $1 \leq i < n$ and the downward-pointing triangle of ℓ_n is adjacent to the upward-pointing triangle of ℓ_1 . The smallest cycle of lozenges is a three-cycle; see Figure 5.7.

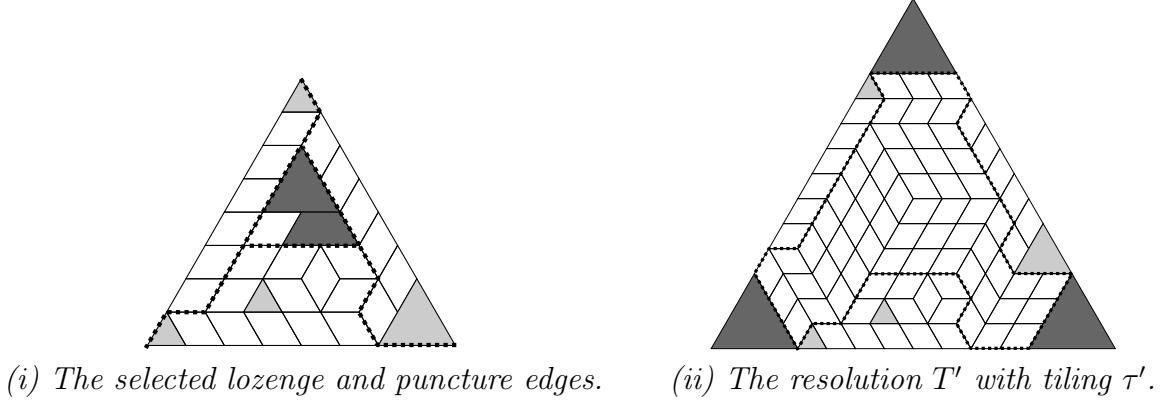
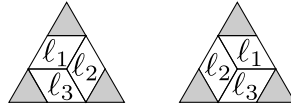
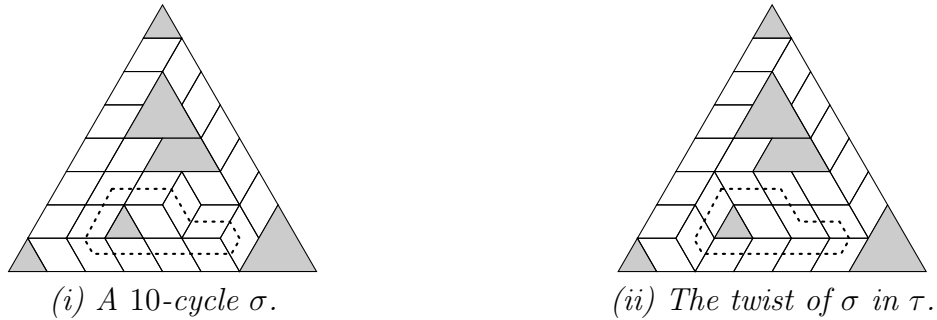


FIGURE 5.6. Resolving overlapping punctures, given the tiling in Figure 3.1.


 FIGURE 5.7. $T_3(x^2, y^2, z^2)$ has two tilings, both are three-cycles of lozenges.

Let $\sigma = \{\ell_1, \dots, \ell_n\}$ be an n -cycle of lozenges in the tiling τ of T . If we replace the lozenges in σ by the n lozenges created by adjoining the downward-pointing triangle of ℓ_i with the upward-pointing triangle of ℓ_{i+1} for $1 \leq i < n$ and the downward-pointing triangle of ℓ_n with the upward-pointing triangle of ℓ_1 , then we get a new tiling τ' of T . We call this new tiling the *twist* of σ in τ . The two three-cycles in Figure 5.7 are twists of each other. See Figure 5.8 for another example of twisting a cycle. A puncture is *inside* the cycle σ if the lozenges of the cycle fully surround the puncture. In Figure 5.8(i), the puncture associated to xy^4z^2 is inside the cycle σ and all other punctures of T are not inside the cycle σ .


 FIGURE 5.8. A 10-cycle σ in the tiling τ (see Figure 3.1) and its twist.

Recall that the perfect matching sign of a tiling τ is denoted by $\text{msgn } \tau$ (see Definition 5.4).

Lemma 5.11. *Let τ be a lozenge tiling of a triangular region $T = T_d(I)$, and let σ be an n -cycle of lozenges in τ . Then the twist τ' of σ in τ satisfies $\text{msgn } \tau' = (-1)^{n-1} \text{msgn } \tau$.*

Proof. Let π and π' be the perfect matching permutations associated to τ and τ' , respectively. Without loss of generality, assume ℓ_i corresponds to the upward- and downward-pointing triangles labeled i . As τ' is a twist of τ by σ , then $\pi'(i) = i + 1$ for $1 \leq i < n$ and $\pi'(n) = 1$. That is, $\pi' = (1, 2, \dots, n) \cdot \pi$, as permutations. Hence, $\text{msgn } \tau' = (-1)^{n-1} \text{msgn } \tau$. \square

5.3.3. Resolutions, cycles of lozenges, and signs.

Resolving a puncture modifies the length of a cycle of lozenges in a predictable fashion. We first need a definition. It uses the starting and end points of lattice paths A_1, \dots, A_m and E_1, \dots, E_m , as introduced at the beginning of Subsection 5.2.

The *E-count* of a cycle is the number of lattice path end points E_j “inside” the cycle. Alternatively, this can be seen as the sum of the side lengths of the non-overlapping punctures plus the sum of the side lengths of the minimal covering regions of pairs of overlapping punctures. For example, the cycles shown in Figure 5.7 have *E*-counts of zero, the cycles shown in Figure 5.8 have *E*-counts of 1, and the (unmarked) cycle going around the outer edge of the tiling shown in Figure 5.8(i) has an *E*-count of $1 + 3 = 4$.

Now we describe the change of a cycle surrounding a puncture when this puncture is resolved.

Lemma 5.12. *Let τ be a lozenge tiling of $T = T_d(I)$, and let σ be an n -cycle of lozenges in τ . Suppose T has a puncture P (or a minimal covering region of a pair of overlapping punctures) with *E*-count k . Let T' be a resolution of P relative to τ . Then the resolution takes σ to an $(n + kl)$ -cycle of lozenges σ' in the resolution, where l is the number of times the splitting chains of the resolution cross the cycle σ in τ . Moreover, l is odd if and only if P is inside σ .*

Proof. Fix a resolution $T' \subset \mathcal{T}_{d+2k}$ of P with tiling τ' as induced by τ .

First, note that if P is a minimal covering region of a pair of overlapping punctures, then any cycle of lozenges must avoid the lozenges present in P as all such lozenges are forcibly chosen, i.e., immutable. Thus, all lozenges of σ are present in τ' .

The resolution takes the cycle σ to a cycle σ' by adding k new lozenges for each unit edge of a lozenge in σ that belongs to a splitting chain. More precisely, such an edge is expanded to $k + 1$ parallel edges. Any two consecutive edges form the opposite sides of a lozenge (see Figure 5.9). Thus, each time a splitting chain of the resolution crosses the cycle σ we insert k new lozenges. As l is the number of times the splitting chains of the resolution cross the cycle σ in τ , the resolution adds exactly kl new lozenges to the extant lozenges of σ . Thus, σ' is an $(n + kl)$ -cycle of lozenges in τ' .

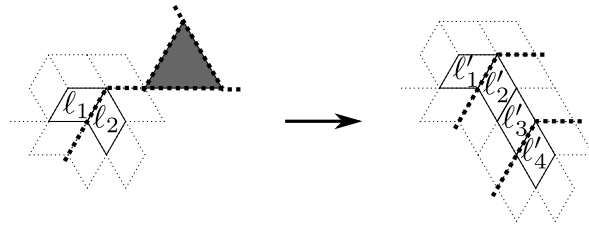


FIGURE 5.9. Expansion of a lozenge cycle at a crossing of a splitting chain.

Since the splitting chains are going from P to the exterior triangle of \mathcal{T}_d , the splitting chains terminate outside the cycle. Hence if the splitting chain crosses into the cycle, it must cross back out. If P is outside σ , then the splitting chains start outside σ , and so l must be even. On the other hand, if P is inside σ , then the splitting chains start inside of σ , and so $l = 3 + 2j$, where j is the number of times the splitting chains cross *into* the cycle. \square

Let τ_1 and τ_2 be tilings of T , and let π_1 and π_2 be their respective perfect matching permutations. Suppose $\pi_2 = \rho\pi_1$, for some permutation ρ . Write ρ as a product of disjoint

cycles whose length is at least two. (Note that these cycles will be of length at least three, as discussed above.) Then each factor corresponds to cycle of lozenges of τ_1 . If all these cycles are twisted we get τ_2 . We call these lozenge cycles the *difference cycles* of τ_1 and τ_2 .

Using the idea of difference cycles, we characterise when two tilings have the same perfect matching sign.

Corollary 5.13. *Let τ be a lozenge tiling of $T = T_d(I)$, and let σ be an n -cycle of lozenges in τ . Then the following statements hold.*

- *The E -count of σ is even if and only if n is odd.*
- *Two lozenge tilings of T have the same perfect matching sign if and only if the sum of the E -counts of the difference cycles is even.*

Proof. Suppose T has a punctures and pairs of overlapping punctures, P_1, \dots, P_a , inside σ that are *not* in a corner, i.e., not associated to x^k, y^k , or z^k , for some k . Let j_i be the E -count of P_i . Similarly, suppose T has b punctures and pairs of overlapping punctures, Q_1, \dots, Q_b , outside σ that are *not* in a corner, i.e., not associated to x^k, y^k , or z^k , for some k . Let k_i be the E -count of Q_i .

If we resolve all of the punctures $P_1, \dots, P_a, Q_1, \dots, Q_b$, then σ is taken to a cycle σ' . By Lemma 5.12, σ' has length

$$n' := n + (j_1 l_1 + \dots + j_a l_a) + (k_1 m_1 + \dots + k_b m_b),$$

where the integers l_1, \dots, l_a are odd and the integers m_1, \dots, m_b are even.

Moreover, the $a + b$ times resolved region T' is, after merging touching punctures, the region of some complete intersection, i.e., of the form $T_e(x^a, y^b, z^c)$, for appropriate values of a, b, c , and e . By [10, Theorem 1.2], every tiling of T' is thus obtained from any other tiling of T' through a sequence of three-cycle twists, as in Figure 5.7. By Lemma 5.11, such twists do not change the perfect matching sign of the tiling, hence n' is an odd integer.

Since n' is odd, $n' - (k_1 m_1 + \dots + k_b m_b) = n + (j_1 l_1 + \dots + j_a l_a)$ is also odd. Thus, n is odd if and only if $j_1 l_1 + \dots + j_a l_a$ is even. Since the integers l_1, \dots, l_a are odd, we see that $j_1 l_1 + \dots + j_a l_a$ is even if and only if an even number of the l_i are odd, i.e., the sum $l_1 + \dots + l_a$ is even. Notice that this sum is the E -count of σ . Thus, claim (i) follows.

Suppose two tilings τ_1 and τ_2 of T have difference cycles $\sigma_1, \dots, \sigma_p$. Then by Lemma 5.11, $\text{msgn } \tau_2 = \text{sgn } \sigma_1 \cdots \text{sgn } \sigma_p \text{msgn } \tau_1$. By claim (i), σ_i is a cycle of odd length if and only if the E -count of σ_i is even. Thus, $\text{sgn } \sigma_1 \cdots \text{sgn } \sigma_p = 1$ if and only if an even number of the σ_i have an odd E -count. An even number of the σ_i have an odd E -count if and only if the sum of the E -counts of $\sigma_1, \dots, \sigma_p$ is even. Hence, claim (ii) follows. \square

Remark 5.14. In [10, Theorem 1.2], Chen, Guo, Jin, and Liu show that three-cycles in the perfect matching permutation correspond to adding or removing boxes from the stacks the tiling represents. See Figure 6.1 where a lozenge tiling of a complete intersection region is shown in correspondence with a plane partition, i.e., boxes stacked in a room.

The lattice path permutation also changes predictably when twisting a cycle of lozenges. To see this we single out certain punctures. We recursively define a puncture of $T \subset \mathcal{T}_d$ to be a *non-floating* puncture if it touches the boundary of \mathcal{T}_d or if it overlaps or touches a non-floating puncture of T . Otherwise we call a puncture a *floating* puncture.

We also distinguish between *preferred* and *acceptable directions* on the splitting chains used for resolving a puncture. Here we use again the perspective of a particle that starts

at a vertex of the puncture and moves on a chain to the corresponding corner vertex of \mathcal{T}_d . Our convention is:

- On the lower-left chain the preferred directions are Southwest and West, the acceptable directions are Northwest and Southeast.
- On the lower-right chain the preferred directions are Southeast and East, the acceptable directions are Northeast and Southwest.
- On the top chain the preferred directions are Northeast and Northwest, the acceptable directions are East and West.

Lemma 5.15. *Let τ be a lozenge tiling of $T = T_d(I)$, and let σ be a cycle of lozenges in τ . Then the lattice path signs of τ and the twist of σ in τ are the same if and only if the E -count of σ is even.*

Proof. Suppose T has n floating punctures. We proceed by induction on n in five steps.

Step 1: The base case.

If $n = 0$, then every tiling of T induces the same bijection $\{A_1, \dots, A_m\} \rightarrow \{E_1, \dots, E_m\}$. Thus, all tilings have the same lattice path sign. Since T has no floating punctures, σ has an E -count of zero. Hence, the claim is true if $n = 0$.

Step 2: The set-up.

Suppose now that $n > 0$, and choose P among the floating punctures and the minimal covering regions of two overlapping floating punctures of T as the one that covers the upward-pointing unit triangle of \mathcal{T}_d with the smallest monomial label. Let $s > 0$ be the side length of P , and let k be the E -count of σ . Furthermore, let v be the lozenge tiling of T obtained as twist of σ in τ . Both, τ and v , induce bijections $\{A_1, \dots, A_m\} \rightarrow \{E_1, \dots, E_m\}$, and we denote by $\lambda \in \mathfrak{S}_m$ and $\mu \in \mathfrak{S}_m$ the corresponding lattice path permutations, respectively. We have to show $\text{lpsgn } \tau = (-1)^k \text{lpsgn } v$, that is,

$$\text{sgn } \lambda = (-1)^k \text{sgn } \mu.$$

Step 3: Resolutions.

We resolve P relative to the tilings τ and v , respectively. For the resolution of P relative to τ , choose the splitting chains so that each unit edge has a preferred direction, except possibly the unit edges on the boundary of a puncture of T ; this is always possible. By our choice of P , no other floating punctures are to the lower-right of P . It follows that no edge on the lower-right chain crosses a lattice path, except possibly at the end of the lattice path.

For the resolution of P relative to v , use the splitting chains described in the previous paragraph, except for the edges that cross the lozenge cycle σ . They have to be adjusted since these unit edges disappear when twisting σ . We replace each such unit edge by a unit edge in an acceptable direction followed by a unit edge in a preferred direction so that the result has the same starting and end point as the unit edge they replace. Note that this is always possible and that this determines the replacement uniquely. The new chains meet the requirements on splitting chains.

Using these splitting chains we resolve the puncture P relative to τ and v , respectively. The result is a triangular region $T' \subset \mathcal{T}_{d+2s}$ with induced tilings τ' and v' , respectively. Denote by σ' the extension of the cycle σ in T' (see Lemma 5.12). Since τ and v differ exactly on the cycle σ and the splitting chains were chosen to be the same except on σ , it follows that twisting σ' in τ' results in the tiling v' of T' .

Step 4: Lattice path permutations.

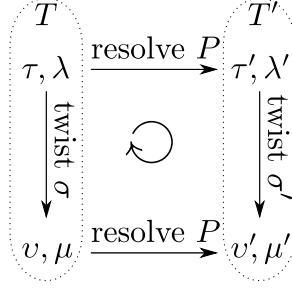


FIGURE 5.10. The commutative diagram used in the proof of Lemma 5.15.

Now we compare the signs of $\lambda, \mu \in \mathfrak{S}_m$ with the signs of λ' and μ' , the lattice path permutations induced by the tilings τ' and v' of T' , respectively.

First, we compare the starting and end points of lattice paths in T and T' . Resolution of the puncture identifies each starting and end point in T with one such point in T' . We refer to these points as the *old* starting and end points in T' . Note that the end points on the puncture P correspond to the end points on the puncture in the Southeast corner of T' . The starting points in T that are on one of the splitting chains used for resolving P relative to τ and v are the same. Assume there are t such points. After resolution, each point gives rise to a new starting and end point in T' . Both are connected by a lattice path that is the same in both resolutions of P . Hence, in order to compare the signs of the permutations λ' and μ' on $m + t$ letters, it is enough to compare the lattice paths between the old starting and end points in both resolutions.

Retain for these points the original labels used in T . Using this labeling, the lattice paths induce permutations $\tilde{\lambda}$ and $\tilde{\mu}$ on m letters. Again, this is the same process in both resolutions. It follows that

$$(5.1) \quad \text{sgn}(\tilde{\lambda}) \cdot \text{lpsgn}(\tau') = \text{sgn}(\tilde{\mu}) \cdot \text{lpsgn}(v').$$

Assume now that P is a puncture. Then the end points on P are indexed by s consecutive integers. Since we retain the labels, the same indices label the end points on the puncture in the Southeast corner of T' . The end points on P correspond to the points in T' whose labels are obtained by multiplying by $x^s y^s$. Consider now the case, where all edges in the lower-right splitting chain in T are in preferred directions. Then the lattice paths induced by τ' connect each point in T' that corresponds to an end point on P to the end point in the Southeast corner of T' with the same index. Thus, $\text{sgn}(\lambda) = \text{sgn}(\tilde{\lambda})$. Next, assume that there is exactly one edge in acceptable direction on the lower-right splitting chain of T . If this direction is Northeast, then the s lattice paths passing through the points in T' corresponding to the end points on P are moved one unit to the North. If the acceptable direction was Southwest, then the edge in this direction leads to a shift of these paths by one unit to the South. In either case, this shift means that the paths in T and T' connect to end points that differ by s transpositions, so $\text{sgn}(\tilde{\lambda}) = (-1)^s \text{sgn}(\lambda)$. More generally, if j is the number of unit edges on the lower-right splitting chain of T that are in acceptable directions, then

$$\text{sgn}(\tilde{\lambda}) = (-1)^{js} \text{sgn}(\lambda).$$

Next, denote by c the number of unit edges on the lower-right splitting chain that have to be adjusted when twisting σ . Since each of these edges is replaced by an edge in a preferred

and one edge in an acceptable direction, after twisting the lower-right splitting chain in T has exactly $j + c$ unit edges in acceptable directions. It follows as above that

$$\operatorname{sgn}(\tilde{\mu}) = (-1)^{(j+c)s} \operatorname{sgn}(\mu).$$

Since a unit edge on the splitting chain has to be adjusted when twisting if and only if it is shared by two consecutive lozenges in the cycle σ , the number c is even if and only if the puncture P is outside σ .

Moreover, as the puncture P has been resolved in T' , we conclude by induction that τ' and v' have the same lattice path sign if and only if the E -count of σ' is even. Thus, we get

$$(5.2) \quad \operatorname{lpsgn}(v') = \begin{cases} (-1)^{k-s} \operatorname{lpsgn}(\tau') & \text{if } P \text{ is inside } \sigma, \\ (-1)^k \operatorname{lpsgn}(\tau') & \text{if } P \text{ is outside } \sigma. \end{cases}$$

Step 5: Bringing it all together.

We consider the two cases separately:

(i) Suppose P is inside σ . Then c is odd. Hence, the above considerations imply

$$\begin{aligned} \operatorname{sgn}(\lambda) &= (-1)^{js} \operatorname{sgn}(\tilde{\lambda}) = (-1)^{js+k-s} \operatorname{sgn}(\tilde{\mu}) \\ &= (-1)^{js+k-s+(j+c)s} \operatorname{sgn}(\mu) \\ &= (-1)^k \operatorname{sgn}(\mu), \end{aligned}$$

as desired.

(ii) Suppose P is outside of σ . Then c is even, and we conclude

$$\begin{aligned} \operatorname{sgn}(\lambda) &= (-1)^{js} \operatorname{sgn}(\tilde{\lambda}) = (-1)^{js+k} \operatorname{sgn}(\tilde{\mu}) \\ &= (-1)^{js+k-s+(j+c)s} \operatorname{sgn}(\mu) \\ &= (-1)^k \operatorname{sgn}(\mu). \end{aligned}$$

Finally, it remains to consider the case where P is the minimal covering region of two overlapping punctures of T . Let \hat{T} be the triangular region that differs from T only by having P as a puncture, and let $\hat{\tau}$ and \hat{v} be the tilings of \hat{T} induced by τ and v , respectively. Since we order the end points of lattice paths using monomial labels, it is possible that the indices of the end points on the Northeast boundary of P in \tilde{T} differ from those of the points on the Northeast boundary of the overlapping punctures in T . However, the lattice paths induced by τ and v connecting the points on the Northeast boundary of P to the points on the Northeast boundary of the overlapping punctures are the same. Hence the lattice paths sign of τ and $\hat{\tau}$ differ in the same ways as the signs of v and \hat{v} . Since we have shown our assertion for $\hat{\tau}$ and \hat{v} , it also follows for τ and v . \square

Using difference cycles, we now characterise when two tilings of a region have the same lattice path sign.

Corollary 5.16. *Let $T = T_d(I)$ be a non-empty, balanced triangular region. Then two tilings of T have the same lattice path sign if and only if the sum of the E -counts (which may count some end points E_j multiple times) of the difference cycles is even.*

Proof. Suppose two tilings τ_1 and τ_2 of T have difference cycles $\sigma_1, \dots, \sigma_p$. By Lemma 5.15, $\operatorname{lpsgn} \tau_1 = \operatorname{lpsgn} \tau_2$ if and only if an even number of the σ_i have an odd E -count. The latter is equivalent to the sum of the E -counts of $\sigma_1, \dots, \sigma_p$ being even. \square

Our above results imply that the two signs that we assigned to a given lozenge tiling, the perfect matching sign (see Definition 5.4) and the lattice path sign (see Definition 5.8), are the same up to a scaling factor depending only on T . The main result of this section follows now easily.

Theorem 5.17. *Let $T = T_d(I)$ be a balanced triangular region. The following statements hold.*

- (i) *Let τ and τ' be two lozenge tilings of T . Then their perfect matching signs are the same if and only if their lattice path signs are the same, that is,*

$$\text{msgn}(\tau) \cdot \text{lpsgn}(\tau) = \text{msgn}(\tau') \cdot \text{lpsgn}(\tau').$$

- (ii) *In particular, we have that*

$$|\det Z(T)| = |\det N(T)|.$$

Proof. Consider two lozenge tilings of T . According to Corollaries 5.13 and 5.16, they have the same perfect matching and the same lattice path signs if and only if the sum of the E -counts of the difference cycles is even. Hence using Theorems 5.5 and 5.9, it follows that $|\det Z(T)| = |\det N(T)|$. \square

This result allows us to move freely between the points of view using lozenge tilings, perfect matchings, and families of non-intersecting lattice paths, as needed. In particular, it implies that rotating a triangular region by 120° or 240° does not change the enumerations. Thus, for example, the three matrices described in Remark 5.10 as well as the matrix given in Example 5.6 all have the same determinant, up to sign.

6. DETERMINANTS

By Theorems 5.5 and 5.9, the enumerations of signed lozenge tilings of a balanced triangular region, where the sign is determined by perfect matchings (see Definitions 5.4) or lattice paths (see Definition 5.8), are both given by determinants of integer matrices. Furthermore, the absolute values of the two determinants are the same by Theorem 5.17. In this section we determine these determinants in various cases. If the determinant is non-vanishing, then we are also interested in its prime divisors, and, failing that, an upper bound on the prime divisors of the enumeration. This is important for applications later on.

6.1. Building enumerations by replacement.

Recall that, by Lemma 3.1, removing a tileable region does not affect unsigned tileability. Using the structure of the bi-adjacency matrix $Z(T)$, we analyse how removing a balanced region affects signed enumerations.

Proposition 6.1. *Let $T \subset \mathcal{T}_d$ be a balanced subregion, and let U be a balanced monomial subregion of T . The following statements hold.*

- (i) $\text{perm } Z(T) = \text{perm } Z(T \setminus U) \cdot \text{perm } Z(U)$; and
- (ii) $|\det Z(T)| = |\det Z(T \setminus U) \cdot \det Z(U)|$.

Proof. Recall that the rows of the matrices $Z(\cdot)$ are indexed by the downward-pointing triangles, and the columns of the matrices $Z(\cdot)$ are indexed by the upward-pointing triangles, using the reverse-lexicographic order of their monomial labels. Reorder the downward-pointing (respectively, upward-pointing) triangles of T so that the triangles of $T \setminus U$ come first and

the triangles of U come second, where we preserve the internal order of the triangles of $T \setminus U$ and U . Using this new ordering, we reorder the rows and columns of $Z(T)$. The result is a block matrix

$$\begin{pmatrix} Z(T \setminus U) & X \\ Y & Z(U) \end{pmatrix}.$$

Since the downward-pointing triangles of U are not adjacent to any upward-pointing triangle of $T \setminus U$, the matrix Y is a zero matrix. Thus, the claims follow by using block matrix formulæ for permanents and determinants. \square

In particular, if we remove a monomial region with a unique lozenge tiling, then we do not modify the enumerations of lozenge tilings in that region. This is true in greater generality.

Proposition 6.2. *Let $T \subset \mathcal{T}_d$ be a balanced subregion, and let U be any subregion of T such that each lozenge tiling of T induces a tiling of U and all the induced tilings of U agree. Then:*

- (i) $Z(T)$ has maximal rank if and only if $Z(T \setminus U)$ has maximal rank.
- (ii) $\text{perm } Z(T) = \text{perm } Z(T \setminus U)$ and $|\det Z(T)| = |\det Z(T \setminus U)|$.

Proof. Part (ii) follows from Theorem 5.5 and Proposition 5.3, and it implies part (i). \square

We point out the following special case.

Corollary 6.3. *Let $T = T_d(I)$ be a balanced triangular region with two punctures P_1 and P_2 that overlap or touch each other. Let P be the minimal covering region of P_1 and P_2 . The following statements hold.*

- (i) $\text{perm } Z(T) = \text{perm } Z(T \setminus P)$; and
- (ii) $|\det Z(T)| = |\det Z(T \setminus P)|$.

Proof. The monomial region $U := P \setminus (P_1 \cup P_2)$ is uniquely tileable. Hence the claims follow from Proposition 6.2 because $T \setminus U = T \setminus P$. \square

We give an example of such a replacement.

Example 6.4. Let $T = T_d(I)$ be a balanced triangular region. Suppose the ideal I has minimal generators $x^{a+\alpha}y^bz^c$ and $x^ay^{b+\beta}z^{c+\gamma}$. The punctures associated to these generators overlap or touch if and only if $a + \alpha + b + \beta + c + \gamma \leq d$. In this case, the minimal overlapping region U of the two punctures is associated to the greatest common divisor $x^ay^bz^c$. Assume that U is not overlapped by any other puncture of T . Then U is uniquely tileable. Hence the regions T and $T' = T \setminus U = T_d(I, x^ay^bz^c)$ have the same enumerations. Note that the ideal $(I, x^ay^bz^c)$ has fewer minimal generators than I . See Corollary 6.6 and Figure 6.2 for an illustration of a special case of splitting a puncture.

The above procedure allows us in some cases to pass from a triangular region to a triangular region with fewer punctures. Enumerations are typically more amenable to explicit evaluations if we have few punctures, as we will see in the next subsection.

6.2. Mahonian determinants.

MacMahon computed the number of plane partitions (finite two-dimensional arrays that weakly decrease in all columns and rows) in an $a \times b \times c$ box as (see, e.g., [53, Page 261])

$$\text{Mac}(a, b, c) := \frac{\mathcal{H}(a)\mathcal{H}(b)\mathcal{H}(c)\mathcal{H}(a+b+c)}{\mathcal{H}(a+b)\mathcal{H}(a+c)\mathcal{H}(b+c)},$$

where a , b , and c are nonnegative integers and $\mathcal{H}(n) := \prod_{i=0}^{n-1} i!$ is the *hyperfactorial* of n . David and Tomei proved in [19] that plane partitions in an $a \times b \times c$ box are in bijection with lozenge tilings in a hexagon with side lengths (a, b, c) , that is, a hexagon whose opposite sides are parallel and have lengths a, b , and c , respectively. However, Propp states on [53, Page 258] that Klarner was likely the first to have observed this. See Figure 6.1 for an illustration of the connection.



FIGURE 6.1. An example of a $2 \times 6 \times 3$ plane partition and the associated lozenge tiling of a hexagon. The grey lozenges are the tops of the boxes.

We use the above formula in many explicit determinantal evaluations. As we are also interested in the prime divisors of the various non-trivial enumerations we consider, we note that $\text{Mac}(a, b, c) > 0$ and the prime divisors of $\text{Mac}(a, b, c)$ are at most $a + b + c - 1$ if a, b , and c are positive. This bound is sharp if $a + b + c - 1$ is a prime number. If one of a, b , or c is zero, then $\text{Mac}(a, b, c) = 1$.

As a first example, we enumerate the (signed) lozenge tilings of a hexagon, i.e., the triangular region of a complete intersection. (See Remark 8.23 for a brief history of results related to the following one.)

Proposition 6.5. *Let a , b , and c be positive integers such that $a \leq b + c$, $b \leq a + c$, and $c \leq a + b$. Suppose that $d = \frac{1}{2}(a + b + c)$ is an integer. Then $T = T_d(x^a, y^b, z^c)$ is a hexagon with side lengths $(d - a, d - b, d - c)$ and*

$$|\det Z(T)| = \text{perm } Z(T) = \text{Mac}(d - a, d - b, d - c).$$

Moreover, the prime divisors of the enumeration are bounded above by $d - 1$.

Proof. As $a \leq b + c$, we have $d = \frac{1}{2}(a + b + c) \geq \frac{1}{2}(a + a) = a$. Similarly, $d \geq b$ and $d \geq c$. Thus T has three punctures of length $d - a$, $d - b$, and $d - c$ in the three corners. Moreover, $d - (d - a + d - b) = d - c$ is the distance between the punctures of length $d - a$ and $d - b$, and similarly for the other two puncture pairings.

Thus, the unit triangles of T form a hexagon with side lengths $(d - a, d - b, d - c)$. By MacMahon's formula we have $\text{perm } Z(T) = \text{Mac}(d - a, d - b, d - c)$. Moreover, each lozenge tiling of T induces the identity permutation as its lattice path permutation. Thus, all tilings of T have the same perfect matching sign by Theorem 5.17. Hence, $|\det N(T)| = |\det Z(T)| = \text{Mac}(d - a, d - b, d - c)$. The prime divisors of this integer are bounded above by $(d - a) + (d - b) + (d - c) - 1 = d - 1$. \square

Combining Example 6.4 and the preceding proposition, we get the following result for a certain region with four punctures.

Corollary 6.6. *Let $T = T_d(x^{a+\alpha}, y^b, z^c, x^a y^\beta z^\gamma)$, where a, b, c , and d are as in Proposition 6.5, α is a positive integer, and β and γ are nonnegative integers, not both zero. Suppose additionally that $\alpha + \beta + \gamma \leq \frac{1}{2}(b + c - a)$. Then*

$$|\det Z(T)| = \text{perm } Z(T) = \text{Mac}(d - a, d - b, d - c),$$

and the prime divisors of the enumeration are bounded above by $d - 1$.

Proof. The assumption $\alpha + \beta + \gamma \leq \frac{1}{2}(b + c - a)$ is equivalent to $a + \alpha + \beta + \gamma \leq d$. Thus, Example 6.4 shows that T has the same enumerations as $T_d(x^a, y^b, z^c)$, and we conclude using Proposition 6.5. See Figure 6.2 for an illustration of the triangular regions. \square

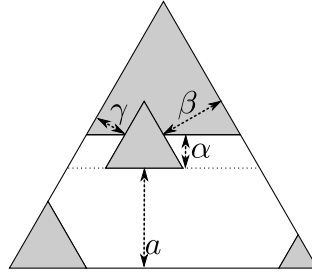


FIGURE 6.2. Covering the punctures associated to $x^{a+\alpha}$ and $x^a y^\beta z^\gamma$ by x^a .

Moreover, combining Propositions 6.1 and 6.5 we get the enumeration for a slightly more complicated triangular region. (We will use this observation in Section 9.) Clearly, the process of removing a hexagon from a puncture can be repeated.

Corollary 6.7. *Let $T = T_d(x^{a+\alpha}, y^b, z^c, x^a y^\beta, x^a z^\gamma)$, where the quadruples (a, b, c, d) and $(\alpha, \beta, \gamma, d - a)$ are both as in Proposition 6.5. In particular, $a + \alpha + \beta + \gamma = b + c$ and $d = \frac{1}{2}(a + b + c)$. Then*

$$|\det Z(T)| = \text{perm } Z(T) = \text{Mac}(d - a, d - b, d - c) \text{Mac}(d - a - \alpha, d - a - \beta, d - a - \gamma),$$

and the prime divisors of the enumeration are bounded above by $d - 1$.

Proof. The region T is obtained from $T_d(x^a, y^b, z^c)$ by replacing the puncture associated to x^a by $T_{d-a}(x^\alpha, y^\beta, z^\gamma)$. See Figure 6.3. \square

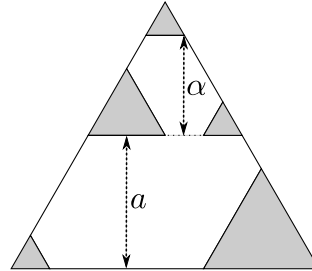


FIGURE 6.3. A simple hexagon with a puncture replaced by a simple hexagon.

A tileable triangular region with punctures in two corners and a third non-corner puncture also has a Mahonian determinant since many of the tiles are fixed.

Proposition 6.8. *Let $T = T_d(x^a, y^b, x^\alpha y^\beta z^{2d-(a+b+\alpha+\beta)})$, where $\alpha + b, a + \beta \leq d \leq a + b$. Then*

$$|\det Z(T)| = \text{perm } Z(T) = \text{Mac}(a + b - d, d - (\alpha + b), d - (a + \beta)),$$

and the prime divisors of the enumeration are sharply bounded above by $d - (\alpha + \beta) - 1$.

Proof. First we note that as $\alpha + b, a + \beta \leq d \leq a + b$, none of the punctures overlap.

Now we consider the families of non-intersecting lattice paths in the lattice $L(T)$. Their end points E_j are *all* along the Northeast boundary of the puncture associated to the monomial $x^\alpha y^\beta z^{2d-(a+b+\alpha+\beta)}$. The lattice paths are thus confined to the region bounded by the starting and end points of the paths. This region is a hexagon of side lengths $(a + b - d, d - (\alpha + b), d - (a + \beta))$. See Figure 6.4 for an illustration. \square

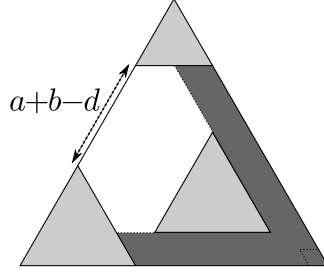


FIGURE 6.4. The portion of T shaded dark grey has fixed tiles of the same orientation, leaving a monomial subregion of T that is a hexagon.

Before computing our final Mahonian-type determinant, we need a more general determinant calculation, which may be of independent interest.

Lemma 6.9. *Let M be an n -by- n matrix with entries*

$$(M)_{i,j} = \begin{cases} \binom{p}{q+j-i} & \text{if } 1 \leq j \leq m, \\ \binom{p}{q+r+j-i} & \text{if } m+1 \leq j \leq n, \end{cases}$$

where p, q, r , and m are nonnegative integers and $1 \leq m \leq n$. Then

$$\det M = \text{Mac}(m, q, r) \text{Mac}(n - m, p - q - r, r) \frac{\mathcal{H}(q+r)\mathcal{H}(p-q)\mathcal{H}(n+r)\mathcal{H}(n+p)}{\mathcal{H}(n+p-q)\mathcal{H}(n+q+r)\mathcal{H}(p)\mathcal{H}(r)}.$$

Proof. We begin by using [13, Equation (12.5)] to evaluate $\det M$ to be

$$\prod_{1 \leq i < j \leq n} (L_j - L_i) \prod_{i=1}^n \frac{(p+i-1)!}{(n+p-L_i)!(L_i-1)!},$$

where

$$L_j = \begin{cases} q+j & \text{if } 1 \leq j \leq m, \\ q+r+j & \text{if } m+1 \leq j \leq n. \end{cases}$$

If we split the products in the previously displayed equation relative to the split in L_j , then we get the following equations:

$$\begin{aligned} \prod_{1 \leq i < j \leq n} (L_j - L_i) &= \left(\prod_{1 \leq i < j \leq m} (j - i) \right) \left(\prod_{m < i < j \leq n} (j - i) \right) \left(\prod_{1 \leq i \leq m < j \leq n} (r + j - i) \right) \\ &= (\mathcal{H}(m)) (\mathcal{H}(n - m)) \left(\frac{\mathcal{H}(n + r) \mathcal{H}(r)}{\mathcal{H}(n + r - m) \mathcal{H}(m + r)} \right) \end{aligned}$$

and

$$\begin{aligned} \prod_{i=1}^n \frac{(p + i - 1)!}{(n + p - L_i)! (L_i - 1)!} &= \left(\prod_{i=1}^n (p + i - 1)! \right) \left(\prod_{i=1}^m \frac{1}{(n + p - q - i)! (q + i - 1)!} \right) \\ &\quad \left(\prod_{i=m+1}^n \frac{1}{(n + p - q - r - i)! (q + r + i - 1)!} \right) \\ &= \left(\frac{\mathcal{H}(n + p)}{\mathcal{H}(p)} \right) \left(\frac{\mathcal{H}(n + p - m - q) \mathcal{H}(q)}{\mathcal{H}(n + p - q) \mathcal{H}(m + q)} \right) \\ &\quad \left(\frac{\mathcal{H}(p - q - r) \mathcal{H}(m + q + r)}{\mathcal{H}(n + p - m - q - r) \mathcal{H}(n + q + r)} \right). \end{aligned}$$

Bringing these equations together we get that $\det M$ is

$$\frac{\mathcal{H}(m) \mathcal{H}(q) \mathcal{H}(r) \mathcal{H}(m + q + r)}{\mathcal{H}(m + r) \mathcal{H}(m + q)} \frac{\mathcal{H}(n - m) \mathcal{H}(p - q - r) \mathcal{H}(n + p - m - q)}{\mathcal{H}(n + r - m) \mathcal{H}(n + p - m - q - r)} \frac{\mathcal{H}(n + r) \mathcal{H}(n + p)}{\mathcal{H}(p) \mathcal{H}(n + p - q) \mathcal{H}(n + q + r)},$$

which, after minor manipulation, yields the claimed result. \square

Remark 6.10. The preceding lemma generalises [38, Lemma 2.2], which handles the case $r = 1$. Furthermore, if $r = 0$, then $\det M = \text{Mac}(n, p - q, q)$, as expected (see the running example, $\det \binom{a+b}{a-i+j}$, in [33]).

A tileable, simply-connected triangular region with four non-floating punctures has a Mahonian-type determinant. This particular region is of interest in Section 9. While in the previous evaluations we directly considered a bi-adjacency matrix, this time we work primarily with a lattice path matrix and then use Theorem 5.17.

Proposition 6.11. *Let $T = T_d(x^a, y^b, z^c, x^\alpha y^\beta)$, where $d = \frac{1}{3}(a + b + c + \alpha + \beta)$ is an integer, $0 < \alpha < a$, $0 < \beta < b$, and $\max\{a, b, c, \alpha + \beta\} \leq d \leq \min\{a + \beta, \alpha + b, a + c, b + c\}$. Then $|\det Z(T)| = \text{perm } Z(T)$ is*

$$\begin{aligned} &\text{Mac}(a + \beta - d, d - a, d - (\alpha + \beta)) \text{Mac}(\alpha + b - d, d - b, d - (\alpha + \beta)) \\ &\times \frac{\mathcal{H}(d - a + d - (\alpha + \beta)) \mathcal{H}(d - b + d - (\alpha + \beta)) \mathcal{H}(d - c + d - (\alpha + \beta)) \mathcal{H}(d)}{\mathcal{H}(a) \mathcal{H}(b) \mathcal{H}(c) \mathcal{H}(d - (\alpha + \beta))}. \end{aligned}$$

Moreover, the prime divisors of the enumeration are bounded above by $d - 1$.

Proof. Note that $\max\{a, b, c, \alpha + \beta\} \leq d$ implies that all four punctures have nonnegative side length. Further, the condition $d \leq \min\{a + \beta, \alpha + b, a + c, b + c\}$ guarantees that none of the punctures overlap.

We now compute the lattice path matrix $N(T)$ as introduced in Subsection 5.2. Recall that a point in the lattice $L(T)$ with label $x^u y^v z^{d-1-(u+v)}$ is identified with the point $(d-1-v, u) \in \mathbb{Z}^2$. Thus, the starting points of the lattice paths are

$$A_i = \begin{cases} (d-b+i-1, d-b+i-1) & \text{if } 1 \leq i \leq \alpha+b-d, \\ (2d-(\alpha+\beta+b)+i-1, 2d-(\alpha+\beta+b)+i-1) & \text{if } \alpha+b-d < i \leq d-c. \end{cases}$$

For the end points of the lattice paths, we get

$$E_j = (c-1+j, j-1), \quad \text{where } 1 \leq j \leq d-c.$$

Thus, the entries of the lattice path matrix $N(T)$ are

$$(N(T))_{i,j} = \begin{cases} \binom{c}{d-b+i-j} & \text{if } 1 \leq i \leq \alpha+b-d, \\ \binom{c}{2d-(\alpha+\beta+b)+i-j} & \text{if } \alpha+b-d < i \leq d-c. \end{cases}$$

Transposing $N(T)$, we get a matrix of the form in Lemma 6.9, where $m = \alpha+b-d$, $n = d-c$, $p = c$, $q = d-b$, and $r = d-(\alpha+\beta)$. Thus, we have the desired determinant evaluation.

Moreover, the only lattice path permutation that admits non-intersecting lattice paths is the identity permutation, so all families of non-intersecting lattice paths have the same sign. Hence $|\det Z(T)| = \text{perm } Z(T)$ by Theorem 5.17.

Finally, as $d-\alpha$ and $d-\beta$ are smaller than d , the prime divisors of $|\det N(T)| = |\det Z(T)|$ are bounded above by $d-1$. \square

Remark 6.12. The evaluation of the determinant in the preceding proposition includes two Mahonian terms and a third non-Mahonian term. It should be noted that both hexagons associated to the Mahonian terms actually show up in the punctured hexagon. See Figure 6.5

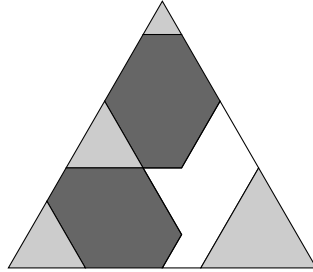


FIGURE 6.5. The darkly shaded hexagons correspond to the two Mahonian terms.

where the darkly shaded hexagons correspond to the Mahonian terms. It is not clear (to us) where the third term comes from, though it may be of interest that if one subtracts $d-(\alpha+\beta)$ from each hyperfactorial parameter, before the evaluation, then what remains is $\text{Mac}(d-a, d-b, d-c)$.

6.3. A single sign.

We exhibit further triangular regions such that all lozenge tilings have the same sign by partially extending Proposition 6.5. Indeed, this is guaranteed to happen if all floating punctures (see the definition preceding Lemma 5.15) have an even side length.

Proposition 6.13. *Let T be a tileable triangular region, and suppose all floating punctures of T have an even side length. Then every lozenge tiling of T has the same perfect matching sign as well as the same lattice path sign, and so $\text{perm } Z(T) = |\det Z(T)|$.*

In particular, simply-connected regions that are tileable have this property.

Proof. The equality of the perfect matching signs follows from Corollary 5.13, and the equality of the lattice path signs from Corollary 5.16. Now Theorem 5.5 implies $\text{perm } Z(T) = |\det Z(T)|$.

The second part is immediate as simply-connected regions have no floating punctures. \square

Remark 6.14. The above proposition vastly extends [10, Theorem 1.2], where hexagons (as in Proposition 6.5) are considered, using a different approach. This special case was also established independently in [31, Section 3.4], with essentially the same proof as [10].

Proposition 6.13 can also be derived from Kasteleyn’s theorem on enumerating perfect matchings [30]. To see this, notice that in the case, where all floating punctures have even side lengths, all “faces” of the bipartite graph $G(T)$ have size congruent to 2 (mod 4).

We now extend Proposition 6.13. To this end we define the *shadow* of a puncture to be the region of T that is both below the puncture and to the right of the line extending from the upper-right edge of the puncture. See Figure 6.6.

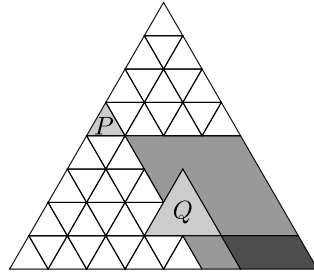


FIGURE 6.6. The puncture P has the puncture Q in its shadow (light grey), but Q does not have a puncture in its shadow (dark grey).

Proposition 6.15. *Let $T = T_d(I)$ be a balanced triangular region. If all floating punctures (and minimal covering regions of overlapping punctures) with other punctures in their shadows have even side length, then any two lozenge tilings of T have the same perfect matching and the same lattice path sign. Thus, $\text{perm } Z(T) = |\det Z(T)|$.*

In particular, $Z(T)$ has maximal rank over a field of characteristic zero if and only if T is tileable.

Proof. Let P be a floating puncture or a minimal covering region with no punctures in its shadow. Then the shadow of P is uniquely tileable, and thus the lozenges in the shadow are fixed in each lozenge tiling of T . Hence, no cycle of lozenges in any tiling of T can contain P . Using Corollary 5.13 and Corollary 5.16, we see that P does not affect the sign of the tilings of T .

Now our assumptions imply that all floating punctures (or minimal covering regions of overlapping punctures) of T that can be contained in a difference cycle of two lozenge tilings of T have even side length. Thus, we conclude $\text{perm } Z(T) = |\det Z(T)|$ as in the proof of Proposition 6.13.

The last assertion follows by Proposition 5.3. \square

6.4. An axes-central puncture.

Last, we consider the case studied by Ciucu, Eisenkölbl, Krattenthaler, and Zare in [13], that is, the case of a hexagon with a central puncture that is equidistant from the hexagon sides along axes through the puncture. We call such a puncture an *axes-central puncture*.

We now describe the ideals whose triangular regions have an axes-central puncture. Let A, B, C , and M be nonnegative integers with at most one of A, B , and C being zero. We must consider two cases, depending on parity.

First, suppose A, B , and C all share the same parity. We form the triangular region

$$T_{A+B+C+M}(x^{B+C+M}, y^{A+C+M}, z^{A+B+M}, x^{\frac{1}{2}(B+C)} y^{\frac{1}{2}(A+C)} z^{\frac{1}{2}(A+B)}).$$

By construction, the region has a puncture of side length A, B , and C in the top, bottom-left, and bottom-right corners, respectively. Further, there is a puncture of side length M that is axes-central. If we let α, β , and γ be the exponents of the mixed term, then we get Figure 6.7(i).

Now suppose A and B differ in parity from C . In this case, the axes-central puncture would have to be located a non-integer distance from the edges of the triangle. To fix this, we shift the puncture up and right one-half unit to create a valid triangular region. In particular, we form the triangular region

$$T_{A+B+C+M}(x^{B+C+M}, y^{A+C+M}, z^{A+B+M}, x^{\frac{1}{2}(B+C+1)} y^{\frac{1}{2}(A+C-1)} z^{\frac{1}{2}(A+B)}).$$

As in the previous case, we get the desired punctures. Moreover, if we let α, β , and γ be the exponents of the mixed term, then we get Figure 6.7(ii).



(i) The parity of C agrees with A and B . (ii) The parity of C differs from A and B .

FIGURE 6.7. The two prototypical figures with axes-central punctures.

Let H be the punctured hexagon with an axes-central puncture associated to the nonnegative integers A, B, C , and M . The total number of lozenge tilings of H as well as a certain (-1) -enumeration of them have been calculated in four theorems (two for each type and parity condition). We recall the four theorems here, although we forgo the exact statements of the enumerations; the explicit enumerations can be found in [13].

Theorem 6.16. [13, Theorems 1, 2, 4, and 5] *Let A, B, C , and M be nonnegative integers, and let H be the associated hexagon with an axes-central puncture. Then the following statements hold:*

- (1) *The number of lozenge tilings of H is $\text{CEKZ}_1(A, B, C, M)$, if A, B , and C share a common parity.*
- (2) *The number of lozenge tilings of H is $\text{CEKZ}_2(A, B, C, M)$, if A, B , and C do not share a common parity.*

- (4) The (-1) -enumeration of lozenge tilings of H is
 - (i) 0, if A, B , and C are all odd;
 - (ii) $\text{CEKZ}_4(A, B, C, M)$, if A, B , and C are all even.
- (5) The (-1) -enumeration of lozenge tilings of H is $\text{CEKZ}_5(A, B, C, M)$, if A, B , and C do not share a common parity.

Moreover, the four functions CEKZ_i are polynomials in M which factor completely into linear terms. Further, each can be expressed as a quotient of products of hyperfactorials and, in each case, the largest hyperfactorial term is $\mathcal{H}(A + B + C + M)$.

Using Proposition 5.3 and Theorem 5.5 together with the fact that the sign used in the (-1) -enumeration in [13] is equivalent to our perfect matching sign, we find the permanent and the determinant of the bi-adjacency matrix of H .

Corollary 6.17. *Let A, B, C , and M be nonnegative integers, and let H be the associated hexagon with an axes-central puncture. Then*

$$\text{perm } Z(H) = \begin{cases} \text{CEKZ}_1(A, B, C, M) & \text{if } A, B, \text{ and } C \text{ share a common parity;} \\ \text{CEKZ}_2(A, B, C, M) & \text{otherwise.} \end{cases}$$

Further, if M is even, then

$$|\det Z(H)| = \begin{cases} \text{CEKZ}_1(A, B, C, M) & \text{if } A, B, \text{ and } C \text{ share a common parity;} \\ \text{CEKZ}_2(A, B, C, M) & \text{otherwise.} \end{cases}$$

And if M is odd, then

$$|\det Z(H)| = \begin{cases} 0 & \text{if } A, B, \text{ and } C \text{ are all odd;} \\ |\text{CEKZ}_4(A, B, C, M)| & \text{if } A, B, \text{ and } C \text{ are all even;} \\ |\text{CEKZ}_5(A, B, C, M)| & \text{otherwise.} \end{cases}$$

Moreover, the prime divisors of the enumerations are bounded above by $A + B + C + M - 1$.

7. MIRROR SYMMETRIC TRIANGULAR REGIONS

A *mirror symmetric* region is a triangular region $T = T_d(I)$ that is invariant under reflection about the vertical line that goes through the top center vertex of the containing triangular region \mathcal{T}_d . Furthermore, we call a puncture an *axial puncture* if its top vertex is on the axis of symmetry, i.e., it is itself symmetric.

In this section we consider mirror symmetric regions under some strong restrictions. We collect the assumptions used for the entirety of the section here.

Assumption 7.1. *Let T be a triangular region that satisfies the following conditions:*

- (i) T is balanced and mirror symmetric.
- (ii) With the exception of a pair of punctures in the bottom corners, all punctures of T are axial punctures.
- (iii) The top-most axial puncture of T is in the top corner of \mathcal{T}_d .
- (iv) No punctures of T touch or overlap.

Recall that such a region $T \subset \mathcal{T}_d$ is balanced if and only if the sum of the side lengths of its punctures equals d because the punctures of T do not overlap. Note that the assumptions (iii) and (iv) above are harmless. Indeed, if one of them is not satisfied, then placing the forced lozenges leads to a mirror symmetric triangular region satisfying (iii) and (iv) (see Corollary 6.3 and Figure 6.4).

Remark 7.2. Each mirror symmetric region satisfying Assumption 7.1 is tileable by lozenges. This follows, for example, from Theorem 3.2.

We need some notation to specify a triangular region T as above. We denote the side length of the base punctures (those in the bottom corners) by b . We label the m axial punctures $1, 2, \dots, m$, starting from the top. The vertical position and the side length of the i^{th} axial puncture are denoted by h_i and d_i , respectively. As the punctures do not touch or overlap, the numbers b and $(h_1, d_1), \dots, (h_m, d_m)$ uniquely define T . See Figure 7.1 for an example.

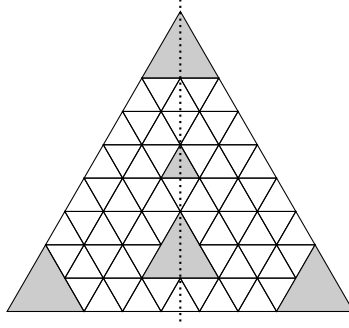


FIGURE 7.1. A mirror symmetric region with corner punctures of length $b = 2$ and axial punctures with parameters $(7, 2)$, $(4, 1)$, and $(1, 2)$.

It is worth recording the conditions the parameters defining T have to satisfy. These parameters also allow us to describe the associated ideal.

Remark 7.3. Let $T \subset \mathcal{T}_d$ be a region with parameters b and $(h_1, d_1), \dots, (h_m, d_m)$. Then:

- (i) Since T is balanced, we have $d := 2b + d_1 + \dots + d_m$.
- (ii) The 1st axial puncture has height $h_1 = d - d_1$ by Assumption 7.1(iii).
- (iii) Assumption 7.1(iv) forces the inequalities $h_i - h_{i+1} > d_{i+1}$ for all $i = 1, \dots, m$.
- (iv) For each $i \in \{1, \dots, m\}$, the integer h_i is even if and only if $d - d_i$ is even because $d - d_i - h_i$ is even by symmetry.
- (v) The region T is $T_d(I)$, where

$$I = (x^{h_1}, y^{d-b}, z^{d-b}, x^{h_2}(yz)^{\frac{1}{2}(d-d_2-h_2)}, \dots, x^{h_m}(yz)^{\frac{1}{2}(d-d_m-h_m)}).$$

7.1. Regularity of the bi-adjacency matrix.

By assumption, the mirror symmetric region T is balanced. Thus, its bi-adjacency matrix $Z(T)$ is a square matrix. Our first main result of this section gives a condition, which implies that $Z(T)$ is not regular.

Theorem 7.4. Let $T = T_d(I)$ be a mirror symmetric region that satisfies Assumption 7.1. If the number of its axial punctures (including the top-most puncture) with odd side length is either 2 or 3 modulo 4, then $\det Z(T) = 0$.

Proof. Instead of directly considering the bi-adjacency matrix $Z(T)$ we study the lattice path matrix $N(T)$. To this end we first turn the region T by 120° . Then $T = T_d(J)$, where

$$J = (x^{d-b}, y^{d-b}, z^{h_1}, (xy)^{\frac{1}{2}(d-d_2-h_2)} z^{h_2}, \dots, (xy)^{\frac{1}{2}(d-d_m-h_m)} z^{h_m}).$$

Note that the axis of symmetry of T passes through its lower right corner. Thus the lattice path matrix $N(T)$ has $d - 2b = d_1 + \dots + d_m$ rows and columns.

Consider a family Λ of non-intersecting lattice paths on $L(T)$. Let λ be the associated permutation, that is, for each $i \in \{1, \dots, d_1 + \dots + d_m\}$, the path starting at the vertex A_i ends at the vertex $E_{\lambda(i)}$.

Recall that a permutation of the form

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix}$$

has signature

$$\text{sgn } \sigma = (-1)^{\lfloor \frac{n}{2} \rfloor}.$$

We refer to σ as a reflection of length n .

Consider now the family of lattice paths $\hat{\Lambda}$ that is obtained from Λ by reflecting it across the axis of symmetry of T . Let $\hat{\lambda}$ be its associated permutation. (Note it is possible that $\Lambda = \hat{\Lambda}$.) This reflection permutes the $d_1 + \dots + d_m$ starting points of the lattice paths and, for each $i = 1, \dots, m$, the d_i end points on the i^{th} axial puncture. It follows that

$$\hat{\lambda} = \alpha \cdot \lambda \cdot \beta,$$

where α is a reflection of length $d_1 + \dots + d_m$ and β is the product of m reflections having lengths d_1, \dots, d_m . Hence, the signature of $\hat{\lambda}$ is

$$\text{sgn } \hat{\lambda} = \text{sgn } \lambda \cdot (-1)^{\lfloor \frac{d_1 + \dots + d_m}{2} \rfloor + \lfloor \frac{d_1}{2} \rfloor + \dots + \lfloor \frac{d_m}{2} \rfloor}.$$

Let q be the number of odd side lengths d_1, \dots, d_m . Then the last formula implies

$$\text{sgn } \hat{\lambda} = \text{sgn } \lambda \cdot (-1)^{\lfloor \frac{q}{2} \rfloor}.$$

Observe that $\lfloor \frac{q}{2} \rfloor$ is odd if and only if q is either 2 or 3 modulo 4.

Finally, suppose that the integer q is either 2 or 3 modulo 4. Then $\text{sgn}(\lambda) = -\text{sgn}(\hat{\lambda})$. Therefore, we have found a bijection from the families of non-intersecting lattice paths of $L(T)$ with positive sign to the families with negative sign. Hence, Theorem 5.9 gives $\det N(T) = 0$. Using Theorem 5.17, we obtain $\det Z(T) = 0$. \square

In particular, the above result applies to the following ideals.

Example 7.5. Consider the ideal

$$I = (x^a, y^c, z^c, x^\alpha y^\gamma z^\gamma),$$

where a, c, α, γ are integers satisfying

$$0 < \alpha < a, \quad 0 < \gamma < c, \quad 2(c - 2\gamma) < 2a - \alpha, \quad \text{and}$$

$$\max\{a, c, \alpha + 2\gamma\} < \frac{1}{3}(a + \alpha + 2c + 2\gamma) \in \mathbb{Z}.$$

Then Theorem 7.4 gives $\det Z(T) = 0$, where $T = T_d(I)$ and $d = \frac{1}{3}(a + \alpha + 2c + 2\gamma)$.

Based on an extensive computer search using *Macaulay2* [40], we offer the following conjectured characterisation of the regularity of the bi-adjacency matrix.

Conjecture 7.6. *Let $T = T_d(I)$ be a mirror symmetric region that satisfies Assumption 7.1. Then $\det Z(T) \neq 0$ if and only if the number of axial punctures (including the top-most puncture) with odd side length is at most one.*

We have additional evidence for this conjecture in the case where the side lengths of all axial punctures except possibly the top axial puncture are even.

Proposition 7.7. *Let $T = T_d(I)$ be a triangular region as in Assumption 7.1 with parameters b and $(h_1, d_1), \dots, (h_m, d_m)$. If d_2, \dots, d_m are even, then $|\det Z(T)| = |\text{perm } Z(T)| > 0$.*

Proof. The assumptions imply that the floating punctures of T all have an even side length. Since T is tileable, the conclusion is an immediate consequence of Proposition 6.13. \square

7.2. Explicit enumerations.

Ciucu [12] gave explicit formulæ for the (unsigned) enumeration of lozenge tilings of mirror symmetric regions. These formulæ were found using techniques first described in [11]. We recall a few definitions following [12, Part B, Section 1].

Let $(a)_k := a(a+1) \cdots (a+k-1)$ be the shifted factorial, also known as the rising factorial. For nonnegative integers m and n , let $B_{m,n}(x)$ and $\overline{B}_{m,n}(x)$ be the monic polynomials defined by

$$(7.1) \quad \begin{aligned} B_{m,n}(x) = & 2^{-mn-m(m-1)/2} (x+n+1)_m (x+n+2)_m \\ & \times \prod_{i=1}^{\lceil \frac{n-1}{2} \rceil} (x+1+i)_{n+1-2i} \times \prod_{i=1}^{\lceil \frac{n}{2} \rceil} \left(x + \frac{1}{2} + i\right)_{n+2-2i} \\ & \times \prod_{i=1}^n \frac{(x+i)_m}{(x+i+1/2)_m} \times \prod_{i=1}^m (2x+n+i+2)_{n+i-1} \end{aligned}$$

and

$$(7.2) \quad \begin{aligned} \overline{B}_{m,n}(x) = & 2^{-mn-n(n+1)/2} (x+m+1)_n \\ & \times \prod_{i=1}^{\lceil \frac{m}{2} \rceil} (x+i)_{m+2-2i} \times \prod_{i=1}^{\lceil \frac{m-1}{2} \rceil} \left(x + \frac{1}{2} + i\right)_{m+1-2i} \\ & \times \prod_{i=1}^m \frac{(x+i)_n}{(x+i+1/2)_n} \times \prod_{i=1}^n (2x+m+i+2)_{m+i}, \end{aligned}$$

respectively.

Moreover, for (possibly empty) sequences $\mathbf{p} = (p_1, \dots, p_m)$ and $\mathbf{q} = (q_1, \dots, q_n)$ of strictly increasing positive integers, define rational numbers $c_{\mathbf{p}, \mathbf{q}}$ and $\bar{c}_{\mathbf{p}, \mathbf{q}}$ by

$$(7.3) \quad c_{\mathbf{p}, \mathbf{q}} = 2^{\binom{n-m}{2} - m} \prod_{i=1}^m \frac{1}{(2p_i)!} \prod_{i=1}^n \frac{1}{(2q_i - 1)!} \frac{\prod_{1 \leq i < j \leq m} (p_j - p_i) \prod_{1 \leq i < j \leq n} (q_j - q_i)}{\prod_{i=1}^m \prod_{j=1}^n (p_i + q_j)}$$

and

$$(7.4) \quad \bar{c}_{\mathbf{p}, \mathbf{q}} = 2^{\binom{n-m}{2} - m} \prod_{i=1}^m \frac{1}{(2p_i - 1)!} \prod_{i=1}^n \frac{1}{(2q_i)!} \frac{\prod_{1 \leq i < j \leq m} (p_j - p_i) \prod_{1 \leq i < j \leq n} (q_j - q_i)}{\prod_{i=1}^m \prod_{j=1}^n (p_i + q_j)},$$

respectively. (We note that in [12], \mathbf{l} is used in place of \mathbf{p} . We changed notation for ease of reading.)

Following still [12, Part B, Section 5], for given parameters \mathbf{p} and \mathbf{q} as above, define polynomials $P_{\mathbf{p},\mathbf{q}}$ and $\overline{P}_{\mathbf{p},\mathbf{q}}$ by

$$(7.5) \quad \begin{aligned} P_{\mathbf{p},\mathbf{q}}(x) = & c_{\mathbf{p},\mathbf{q}} B_{m,n}(x + p_m - m) \\ & \times \prod_{i=1}^m \prod_{j=i}^{p_i-1} (x + p_m - j)(x + p_m - m + n + j + 2) \\ & \times \prod_{i=1}^n \prod_{j=1}^{q_i-1} (x + p_m - m + n - j + 1)(x + p_m + j + 1) \end{aligned}$$

and

$$(7.6) \quad \begin{aligned} \overline{P}_{\mathbf{p},\mathbf{q}}(x) = & \overline{c}_{\mathbf{p},\mathbf{q}} \overline{B}_{m,n}(x + p_m - m) \\ & \times \prod_{i=1}^m \prod_{j=i}^{p_i-1} (x + p_m - j)(x + p_m - m + n + j + 1) \\ & \times \prod_{i=1}^n \prod_{j=1}^{q_i-1} (x + p_m - m + n - j)(x + p_m + j + 1), \end{aligned}$$

respectively. Furthermore, define the following modifications of $\mathbf{p} = (p_1, \dots, p_m)$:

$$\mathbf{p} - 1 = \begin{cases} (p_1 - 1, \dots, p_m - 1) & \text{if } p_1 > 1 \\ (p_2 - 1, \dots, p_m - 1) & \text{if } p_1 = 1 \end{cases}.$$

and

$$\mathbf{p}^{(m)} = (p_1, \dots, p_{m-1}).$$

Let $T = T_d(I)$ be a mirror symmetric region as in Assumption 7.1 with parameters b and $(h_1, d_1), \dots, (h_s, d_s)$, where $s \geq 1$ and d_2, \dots, d_{s-1} are all even. In order to use the notation introduced above, define $a := d_1$ and $k := d_2 + d_3 + \dots + d_s$. Moreover, if d_s is even (i.e., k is even), then set

$$\begin{aligned} \mathbf{p} := & \left(1, 2, \dots, \left\lfloor \frac{h_s}{2} \right\rfloor, \left\lfloor \frac{d_s + h_s}{2} \right\rfloor + 1, \left\lfloor \frac{d_s + h_s}{2} \right\rfloor + 2, \dots, \left\lfloor \frac{h_{s-1}}{2} \right\rfloor, \dots, \right. \\ & \left. \left\lfloor \frac{d_3 + h_3}{2} \right\rfloor + 1, \left\lfloor \frac{d_3 + h_3}{2} \right\rfloor + 2, \dots, \left\lfloor \frac{h_2}{2} \right\rfloor, \left\lfloor \frac{d_2 + h_2}{2} \right\rfloor + 1, \left\lfloor \frac{d_2 + h_2}{2} \right\rfloor + 2, \dots, \left\lceil \frac{h_1}{2} \right\rceil \right) \end{aligned}$$

and $\mathbf{q} = \emptyset$. More precisely, \mathbf{p} is a concatenation of s lists, where the i^{th} list, counted from the end, consists of consecutive integers $\left\lfloor \frac{d_i + h_i}{2} \right\rfloor + 1, \left\lfloor \frac{d_i + h_i}{2} \right\rfloor + 2, \dots, \left\lfloor \frac{h_{i-1}}{2} \right\rfloor$ if $2 \leq i \leq s-1$.

Thus, if $s = 1$, then $\mathbf{p} = (1, 2, \dots, \left\lceil \frac{h_1}{2} \right\rceil)$. If $s \geq 2$, then \mathbf{p} has

$$m = \left\lfloor \frac{h_s}{2} \right\rfloor + \left\lceil \frac{h_1}{2} \right\rceil - \left\lfloor \frac{d_2 + h_2}{2} \right\rfloor + \sum_{i=2}^{s-1} \left(\left\lfloor \frac{h_{i-1}}{2} \right\rfloor - \left\lfloor \frac{d_i + h_i}{2} \right\rfloor \right)$$

entries.

If d_s is odd (i.e., k is odd), then set $\mathbf{p} := (1, 2, \dots, \lfloor \frac{h_s}{2} \rfloor)$ and

$$\mathbf{q} := \left(\left\lfloor \frac{d_s}{2} \right\rfloor + 1, \left\lfloor \frac{d_s}{2} \right\rfloor + 2, \dots, \left\lfloor \frac{h_{s-1} - h_s}{2} \right\rfloor, \right. \\ \left. \left\lfloor \frac{d_{s-1} + h_{s-1} - h_s}{2} \right\rfloor + 1, \left\lfloor \frac{d_{s-1} + h_{s-1} - h_s}{2} \right\rfloor + 2, \dots, \left\lfloor \frac{h_{s-2} - h_s}{2} \right\rfloor, \dots, \right. \\ \left. \left\lfloor \frac{d_2 + h_2 - h_s}{2} \right\rfloor + 1, \left\lfloor \frac{d_2 + h_2 - h_s}{2} \right\rfloor + 2, \dots, \left\lfloor \frac{h_1 - h_s}{2} \right\rfloor \right).$$

This time the $(i-1)^{\text{st}}$ sublist of \mathbf{q} , counted from the end, consists of consecutive integers $\lfloor \frac{d_i + h_i - h_s}{2} \rfloor + 1, \lfloor \frac{d_i + h_i - h_s}{2} \rfloor + 2, \dots, \lfloor \frac{h_{i-1} - h_s}{2} \rfloor$ if $2 \leq i \leq s$. Thus, if $s = 1$, then $\mathbf{q} = \emptyset$. If $s \geq 2$, then \mathbf{q} has

$$n = \left\lfloor \frac{h_{s-1} - h_s}{2} \right\rfloor - \left\lfloor \frac{d_s}{2} \right\rfloor + \sum_{i=2}^s \left(\left\lfloor \frac{h_{i-1} - h_s}{2} \right\rfloor - \left\lfloor \frac{d_i + h_i - h_s}{2} \right\rfloor \right)$$

entries.

Using this notation, we can now apply a result in [12] that enumerates the number of unsigned lozenge tilings of a mirror-symmetric region in various cases.

Theorem 7.8. *Let $T = T_d(I)$ be a mirror symmetric satisfying Assumption 7.1 with parameters b and $(h_1, d_1), \dots, (h_s, d_s)$, where $s \geq 1$ and d_2, \dots, d_{s-1} are all even. Define a , k , \mathbf{p} , and \mathbf{q} be as above. Then:*

- (i) *If d_1 is even, d_s is even, and $h_s \geq 2$, then $\text{perm } Z(T) = 2^m P_{\emptyset, \mathbf{p}} \left(\frac{a+k-2}{2} \right) \overline{P}_{\mathbf{p}-1, \emptyset} \left(\frac{a}{2} \right)$.*
- (ii) *If d_1 is even, d_s is even, and $0 \leq h_s < 2$, then $\text{perm } Z(T) = 2^m P_{\emptyset, \mathbf{p}} \left(\frac{a+k-2}{2} \right) P_{\mathbf{p}-1, \emptyset} \left(\frac{a}{2} \right)$.*
- (iii) *If d_1 is odd and d_s is even, then $\text{perm } Z(T) = 2^m P_{\emptyset, \mathbf{p}^{(m)}} \left(\frac{a+k-1}{2} \right) \overline{P}_{\mathbf{p}, \emptyset} \left(\frac{a-1}{2} \right)$.*
- (iv) *If d_1 is even, d_s is odd, and $h_s \geq 2$, then $\text{perm } Z(T) = 2^{m+n} \overline{P}_{\mathbf{p}, \mathbf{q}} \left(\frac{a+k-1}{2} \right) P_{\mathbf{q}, \mathbf{p}^{(m)}} \left(\frac{a}{2} \right)$.*
- (v) *If d_1 and d_s are both odd, then $\text{perm } Z(T) = 2^{m+n} \overline{P}_{\mathbf{p}, \mathbf{q}^{(n)}} \left(\frac{a+k}{2} \right) P_{\mathbf{q}, \mathbf{p}} \left(\frac{a-1}{2} \right)$.*

Moreover, in cases (i)–(iii), $|\det Z(T)| = \text{perm } Z(T)$.

Proof. By Proposition 5.3, we know that $\text{perm } Z(T)$ enumerates the *unsigned* lozenge tilings of T . Furthermore, observe that the five conditions in the statement are equivalent to the following conditions on a , k , and \mathbf{p} in the corresponding order:

- (i) k is even, a is even, and $p_1 = 1$.
- (ii) k is even, a is even, and $p_1 > 1$.
- (iii) k is even and a is odd.
- (iv) k is odd, a is even, and $\mathbf{p} \neq \emptyset$.
- (v) k and a are both odd.

Now the stated formulæ for $\text{perm } Z(T)$ follow from [12, Part B, Theorem 1.1].

Finally, note that k is even if and only if all but the top axial punctures have even side length. Hence Proposition 6.13 gives $|\det Z| = \text{perm } Z$ in the cases (i)–(iii). \square

In case (v) of Theorem 7.8, the determinant is not equal to the permanent. This is also true in case (iv) in general.

Remark 7.9. In case (v) of Theorem 7.8, as d_1 and d_s are odd, $T_d(I)$ has exactly two axial punctures with odd side lengths. Thus, Theorem 7.4 gives $\det Z(T_d(I)) = 0 < \text{perm } Z(T)$.

Moreover, in case (iv), in general, the permanent does not enumerate the *signed* lozenge tilings of $T_d(I)$. Consider, for example, the ideal $I = (x^5, y^5, z^5, x^2y^2z^2)$. Then $|\det Z(T_7(I))| = 50$ and $\text{perm } Z(T_7(I)) = 54$.

Observe that in Theorem 7.8(iv), the conditions that d_1 be even and d_s be odd force that h_s is even. Thus, we left out precisely the case, where $h_s = 0$ although this case is included in [12]. It seems (to us) that in this specific situation the formula given in [12, Part B, Theorem 1.1] needs an adjustment.

Example 7.10. Let $T = T_5(I)$, where $I = (x^3, y^4, z^4, y^2z^2)$; see Figure 7.2. Then $b = 1$,

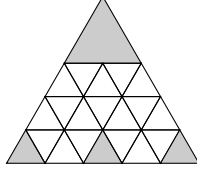


FIGURE 7.2. The triangular region $T_5(x^3, y^4, z^4, y^2z^2)$.

$(h_1, d_1) = (3, 2)$, and $(h_2, d_2) = (0, 1)$. That is, $a = 2$, $b = 1$, $k = 1$, $\mathbf{p} = \emptyset$, and $\mathbf{q} = \{1\}$. If we apply the formula in Theorem 7.8(iv) with these parameters, then we get

$$2^1 \cdot \overline{P}_{\emptyset, \{1\}}(1) \cdot P_{\{1\}, \emptyset}(1) = 2 \cdot 2 \cdot 3 = 12.$$

However, in this case we can use Proposition 6.11 with parameters $c = 3$, $a = b = 4$, and $\alpha = \beta = 2$ to see that

$$\text{perm } Z(T) = \text{Mac}(1, 1, 1) \cdot \text{Mac}(1, 1, 1) \cdot \frac{\mathcal{H}(2)\mathcal{H}(2)\mathcal{H}(3)\mathcal{H}(5)}{\mathcal{H}(4)\mathcal{H}(4)\mathcal{H}(3)\mathcal{H}(1)} = 2 \cdot 2 \cdot \frac{1 \cdot 1 \cdot 2 \cdot 288}{12 \cdot 12 \cdot 2 \cdot 1} = 8.$$

Similarly, we can compute the determinant of the lattice path matrix of T :

$$\det N(T) = \det \begin{pmatrix} 4 & 2 \\ 6 & 1 \end{pmatrix} = -8.$$

We note, however, that if modify the formula in Theorem 7.8(iv) to read (notice the function P is now the function \overline{P})

$$2^{m+n} \overline{P}_{\emptyset, \mathbf{q}} \left(\frac{a+k-1}{2} \right) \overline{P}_{\mathbf{q}, \emptyset} \left(\frac{a}{2} \right),$$

then we *do* get the correct value:

$$2^1 \cdot \overline{P}_{\emptyset, \{1\}}(1) \cdot \overline{P}_{\{1\}, \emptyset}(1) = 2 \cdot 2 \cdot 2 = 8.$$

The above modification of the formula in Theorem 7.8(iv) gives the correct result in general. It follows that we get an explicit formula for the permanent, and hence for the determinant of the bi-adjacency matrix of the associated bipartite graph.

Theorem 7.11. *Let $T = T_d(I)$ be a triangular region as in Assumption 7.1 with parameters b and $(h_1, d_1), \dots, (h_s, d_s)$, where $s \geq 1$ and d_2, \dots, d_{s-1} are all even. Define a , k , \mathbf{p} , and \mathbf{q} as introduced above Theorem 7.8.*

If d_1 is even, d_s is odd, and $h_s = 0$, then

$$|\det Z(T)| = \text{perm } Z(T) = 2^{m+n} \overline{P}_{\emptyset, \mathbf{q}} \left(\frac{a+k-1}{2} \right) \overline{P}_{\mathbf{q}, \emptyset} \left(\frac{a}{2} \right).$$

Proof. We first note that d_1 being even implies a is even, d_s being odd implies k is odd, and $h_s = 0$ implies $\mathbf{p} = \emptyset$. Since d_2, \dots, d_{s-1} are all even, all floating punctures of $Z(T)$ have even side length. Hence, Proposition 6.13 gives $|\det Z(T)| = \text{perm } Z(T)$.

Proposition 5.3 shows that $\text{perm } Z(T)$ enumerates the *unsigned* lozenge tilings of T . Thus, by making a single adjustment, the claim follows as in [12, Part B, Section 3, Proof of Theorem 1.1]. We defer the details to the Appendix (see Proposition A.1) as this requires different arguments that will not be used again in the body of this work. \square

The explicit formulæ above allow us to extract upper bounds on the prime divisors of the signed enumerations of lozenge tilings. In the next section we will show how this is related to the presence of the weak Lefschetz property in positive characteristic.

Proposition 7.12. *Let $T = T_d(I)$ be a triangular region as in Assumption 7.1 with parameters b and $(h_1, d_1), \dots, (h_s, d_s)$, where $s \geq 1$ and d_2, \dots, d_{s-1} are all even.*

If either (i) d_s is even, or (ii) d_1 is even, d_s is odd, and $h_s = 0$, then the prime divisors of $\det Z(T)$ are less than d .

Proof. Each of the equations (7.1)–(7.6) is given as a product of factors, so the prime divisors are bounded by the largest factors. In particular, $B_{m,n}(x)$ and $\overline{B}_{m,n}(x)$, given in (7.1) and (7.2), respectively, have largest factors bounded above by $2x + 2n + 2m$. Similarly, $c_{\mathbf{p},\mathbf{q}}$ and $\overline{c}_{\mathbf{p},\mathbf{q}}$, given in (7.3) and (7.4), respectively, have largest factors bounded by $\max\{p_m - p_1, q_n - q_1\}$. Bringing this together, along with the factors in their defining equations, we see that $P_{\mathbf{p},\mathbf{q}}(x)$ and $\overline{P}_{\mathbf{p},\mathbf{q}}(x)$, given in (7.5) and (7.6), respectively, have largest factors bounded above by $\max\{2x + 2p_m + 2n, x + p_m + q_n\}$.

By definition, $q_n \leq p_m = \lceil \frac{h_1}{2} \rceil$, and since $h_1 \leq 2b + k - 2$, we have that $q_n \leq p_m < b + \frac{1}{2}k$. Further, m and n are bounded above by the number of vertebra in T less the number of vertebra covered by axial punctures, that is m and n are bounded above by $(b + \lfloor \frac{k}{2} \rfloor) - \lfloor \frac{k}{2} \rfloor = b$.

In each of the four relevant formulæ, see Theorems 7.8(i)–(iii) and 7.11, one of \mathbf{p} and \mathbf{q} is empty. In any case, the largest factor of the first term of all four formulae is bounded by $2x + 2\max\{m, n\}$, where x is bounded by $\frac{1}{2}(a + k - 1)$, hence it is bounded by $a + k - 1 + 2b = d - 1$. Similarly, in any case, the largest factor of the second term of all four formulæ is bounded by $2x + 2\max\{p_m, q_n\}$, where x is bounded by $\frac{1}{2}a$, hence it is bounded by $a + 2b + k - 2 < d - 1$. \square

The explicit signed enumerations in Theorems 7.8 and 7.11 were found by a decomposition of T into two regions which could be enumerated separately using techniques first described in [11]. Can this approach be used to handle the, conjecturally, one remaining case with non-zero determinant?

Question 7.13. Let $T = T_d(I)$ be a triangular region as in Assumption 7.1 with parameters b and $(h_1, d_1), \dots, (h_s, d_s)$. Suppose d_1 is even and exactly one of d_2, \dots, d_s is odd. Is then the signed enumeration of lozenge tilings of T not zero, i.e., is $\det Z(T) \neq 0$? If so, what is the explicit enumeration thereof?

Note that a positive answer to the first of the preceding questions would prove one direction of Conjecture 7.6.

8. THE WEAK LEFSCHETZ PROPERTY

Let $R = K[x_1, \dots, x_n]$ be the standard graded n -variate polynomial ring over an infinite field K , and let A be a standard graded quotient of R . Suppose A is Artinian, i.e., A is finite

dimensional as a vector space over K . Then A is said to have the *weak Lefschetz property* if there exists a linear form $\ell \in [A]_1$ such that, for all integers d , the multiplication map $\times \ell : [A]_d \rightarrow [A]_{d+1}$ has maximal rank, that is, the map is injective or surjective. Such a linear form is called a *Lefschetz element* of A .

In what follows, we first derive a few general tools for determining the presence or absence of the weak Lefschetz property. Using these results we find a more specific criterion for the weak Lefschetz property for monomial ideals in $K[x, y, z]$. In particular, we relate the bi-adjacency and lattice path matrices to the maps that decide the weak Lefschetz properties. We show that the prime divisors of the signed enumerations of lozenge tilings of the triangular region to an ideal govern the presence or absence of the weak Lefschetz property of the ideal. We close this section with a reinterpretation of a few of the results in the previous sections.

8.1. Tools.

There are some general results that are helpful in order to determine the presence or absence of the weak Lefschetz property. We recall or derive these tools here.

First, we review the appropriate generalisations of some concepts that were first discussed in Subsection 2.4. All R -modules in this paper are assumed to be finitely generated and graded. Let M be an Artinian R -module. The *socle* of M , denoted $\text{soc } M$, is the annihilator of $\mathfrak{m} = (x_1, \dots, x_n)$, the homogeneous maximal ideal of R , that is, $\text{soc } M = \{y \in M \mid y \cdot \mathfrak{m} = 0\}$. The *socle degree* or *Castelnuovo-Mumford regularity* of M is the maximum degree of a non-zero element in $\text{soc } M$. The module M is said to be *level* if all socle generators have the same degree, i.e., its socle is concentrated in one degree.

Alternatively, assume that the minimal free resolution of M over R ends with a free module $\bigoplus_{i=1}^m R(-t_i)^{r_i}$, where $0 < t_1 < \dots < t_m$ and $0 < r_i$ for all i . In this case, the socle generators of M have degrees $t_1 - n, \dots, t_m - n$. Thus, M is level if and only if $m = 1$.

We now begin with deriving some rather general facts. Once multiplication by a general linear form on an algebra is surjective, then it remains surjective.

Proposition 8.1. [45, Proposition 2.1(a)] *Let $A = R/I$ be an Artinian standard graded K -algebra, and let ℓ be a general linear form. If the map $\times \ell : [A]_d \rightarrow [A]_{d+1}$ is surjective for some $d \geq 0$, then $\times \ell : [A]_{d+1} \rightarrow [A]_{d+2}$ is surjective.*

This can be extended to modules generated in degrees that are sufficiently small.

Lemma 8.2. *Let M be a graded R -module such that the degrees of its minimal generators are at most d . Let $\ell \in R$ be a general linear form. If the map $\times \ell : [M]_{d-1} \rightarrow [M]_d$ is surjective, then the map $\times \ell : [M]_{j-1} \rightarrow [M]_j$ is surjective for all $j \geq d$.*

Proof. Consider the exact sequence $[M]_{d-1} \xrightarrow{\times \ell} [M]_d \rightarrow [M/\ell M]_d \rightarrow 0$. Notice the first map is surjective if and only if $[M/\ell M]_d = 0$. Thus, the assumption gives $[M/\ell M]_d = 0$. Hence $[M/\ell M]_{j+1}$ is zero for all $j \geq d$ because M does not have minimal generators having a degree greater than d , by assumption. \square

As a consequence, we get a generalisation of [45, Proposition 2.1(b)], which considers the case of level algebras.

Corollary 8.3. *Let M be an Artinian graded R -module such that the degrees of its non-trivial socle elements are at least $\geq d - 1$. Let $\ell \in R$ be a general linear form. If the map $\times \ell : [A]_{d-1} \rightarrow [A]_d$ is injective, then the map $\times \ell : [A]_{j-1} \rightarrow [A]_j$ is injective for all $j \leq d$.*

Proof. The K -dual of M is $M^\vee = \text{Hom}_K(M, K)$. Then $\times \ell : [M]_{j-1} \rightarrow [M]_j$ is injective if and only if the map $\times \ell : [M^\vee]_{-j} \rightarrow [M^\vee]_{-j+1}$ is surjective. The assumption on the socle of M means that the degrees of the minimal generators of M^\vee are at most $-d + 1$. Thus, we conclude by Lemma 8.2. \square

The above observations imply that to decide the presence of the weak Lefschetz property we need only check near a “peak” of the Hilbert function.

Proposition 8.4. *Let $A \neq 0$ be an Artinian standard graded K -algebra. Let ℓ be a general linear form. Then:*

- (i) *Let d be the smallest integer such that $h_A(d-1) > h_A(d)$. If A has a non-zero socle element of degree less than $d-1$, then A does not have the weak Lefschetz property.*
- (ii) *Let d be the largest integer such that $h_A(d-2) < h_A(d-1)$. If A has the weak Lefschetz property, then*
 - (a) $\times \ell : [A]_{d-2} \rightarrow [A]_{d-1}$ *is injective,*
 - (b) $\times \ell : [A]_{d-1} \rightarrow [A]_d$ *is surjective, and*
 - (c) *A has no socle generators of degree less than $d-1$.*
- (iii) *Let $d \geq 0$ be an integer such that A has the following three properties:*
 - (a) $\times \ell : [A]_{d-2} \rightarrow [A]_{d-1}$ *is injective,*
 - (b) $\times \ell : [A]_{d-1} \rightarrow [A]_d$ *is surjective, and*
 - (c) *A has no socle generators of degree less than $d-2$.*

Then A has the weak Lefschetz property.

Proof. Suppose in case (i) A has a socle element $y \neq 0$ of degree $e < d-1$. Then $\ell y = 0$, and so the map $\times \ell : [A]_e \rightarrow [A]_{e+1}$ is not injective. Moreover, since $e < d-1$ we have $h_{R/I}(e) \leq h_{R/I}(e+1)$. Hence, the map $\times \ell : [A]_e \rightarrow [A]_{e+1}$ does not have maximal rank. This proves claim (i).

For showing (ii), suppose A has the weak Lefschetz property. Then, by its definition, A satisfies (ii)(a) and (ii)(b) because $h_A(d-1) \geq h_A(d)$. Assume (ii)(c) is not true, that is, A has a socle element $y \neq 0$ of degree $e < d-1$. Then the map $\times \ell : [A]_e \rightarrow [A]_{e+1}$ is not injective. Since A has the weak Lefschetz property, this implies $h_A(e) > h_A(e+1)$. Hence the assumption on d gives $e \leq d-3$. However, this means that the Hilbert function of A is not unimodal. This is impossible if A has the weak Lefschetz property (see [28]).

Finally, we prove (iii). Corollary 8.3 and Assumptions (iii)(a), and (iii)(c) imply that the map $\times \ell : [A]_{i-2} \rightarrow [A]_{i-1}$ is injective if $i \leq d$. Furthermore, using (iii)(b) and Proposition 8.1, we see that $\times \ell : [A]_{i-1} \rightarrow [A]_i$ is surjective if $i \geq d$. Thus, A has the weak Lefschetz property. \square

The same arguments also give the following result.

Corollary 8.5. *Let A be an Artinian standard graded K -algebra, and let ℓ be a general linear form. Suppose there is an integer d such that $0 \neq h_A(d-1) = h_A(d)$ and A has no socle elements of degree less than $d-1$. Then A has the weak Lefschetz property if and only if $\times \ell : [A]_{d-1} \rightarrow [A]_d$ is bijective.*

Sometimes we will rephrase the above assumption on the Hilbert function of A by saying that it has “twin peaks.”

The following easy, but useful observation is essentially the content of [45, Proposition 2.2].

Proposition 8.6. *Let $A = R/I$ be an Artinian K -algebra, where I is generated by monomials and K is an infinite field. Let d and $e > 0$ be integers. Then the following conditions are equivalent:*

- (i) *The multiplication map $\times L^e : [A]_{d-e} \rightarrow [A]_d$ has maximal rank, where $L \in R$ is a general linear form.*
- (ii) *The multiplication map $\times (x_1 + \cdots + x_n)^e : [A]_{d-e} \rightarrow [A]_d$ has maximal rank.*

Proof. For the convenience of the reader we recall the argument. Let $L = a_1x_1 + \cdots + a_rx_r \in R$ be a general linear form. Thus, we may assume that each coefficient a_i is not zero. Rescaling the variables x_i such that L becomes $x_1 + \cdots + x_n$ provides an automorphism of R that maps I onto I . \square

Hence, for monomial algebras, it is enough to decide whether the sum of the variables is a Lefschetz element. As a consequence, we show that, for a monomial algebra, the presence of the weak Lefschetz property in characteristic zero is equivalent to the presence of the weak Lefschetz property in some (actually, almost every) positive characteristic. Here we use that the minimal generators of a monomial ideal are not affected by the characteristic of the ground field K .

Recall that a *maximal minor* of a matrix B is the determinant of a maximal square submatrix of B . Let us also mention again that throughout this section we assume that K is an infinite field.

Corollary 8.7. *Let A be an Artinian monomial K -algebra. Then the following conditions are equivalent:*

- (i) *A has the weak Lefschetz property in characteristic zero.*
- (ii) *A has the weak Lefschetz property in some positive characteristic.*
- (iii) *A has the weak Lefschetz property in every sufficiently large positive characteristic.*

Proof. Let $\ell = x_0 + \cdots + x_n$. By Proposition 8.6, A has the weak Lefschetz property if, for each integer d , the map $\times \ell : [A]_{d-1} \rightarrow [A]_d$ has maximal rank. As A is Artinian, there are only finitely many non-zero maps to be checked. Fixing monomial bases for all non-trivial components $[A]_j$, the mentioned multiplication maps are described by zero-one matrices.

Suppose A has the weak Lefschetz property in some characteristic $q \geq 0$. Then for each of the finitely many matrices above, there exists a maximal minor that is non-zero in K , hence non-zero as an integer. The finitely many non-zero maximal minors, considered as integers, have finitely many prime divisors. Hence, there are only finitely many prime numbers, which divide one of these minors. If the characteristic of K does not belong to this set of prime numbers, then A has the weak Lefschetz property. \square

The following result is a generalisation of [38, Proposition 3.7].

Lemma 8.8. *Let A be an Artinian monomial K -algebra. Suppose that a is the least positive integer such that $x_i^a \in I$ whenever $1 \leq i \leq n$, and suppose that the Hilbert function of A weakly increases to degree s . Then, for any positive prime p such that $a \leq p^m \leq s$ for some positive integer m , A fails to have the weak Lefschetz property in characteristic p .*

Proof. Suppose the characteristic of K is p , and let $\ell = x_1 + \cdots + x_n$. Then, by the Frobenius endomorphism, $\ell \cdot \ell^{p^m-1} = \ell^{p^m} = x_1^{p^m} + \cdots + x_n^{p^m}$. Moreover, $\ell^{p^m} = 0$ in A as $a \leq p^m$. Since $\ell \neq 0$ in A , the map $\times \ell^{p^m-1} : [A]_1 \rightarrow [A]_{p^m}$ is not injective. Thus, A does not have the weak Lefschetz property by Proposition 8.6. \square

We conclude this subsection by noting that any Artinian ideal in two variables has the weak Lefschetz property. This was first proven for characteristic zero in [28, Proposition 4.4] and then for arbitrary characteristic in [48, Corollary 7], though it was not specifically stated therein (see [38, Remark 2.6]). We provide a brief, direct proof of this fact to illustrate the weak Lefschetz property. Unfortunately, the argument cannot be extended to the case of three variables, not even for monomial ideals.

Proposition 8.9. *Let $R = K[x, y]$, where K is an infinite field of arbitrary characteristic. Then every Artinian graded algebra R/I has the weak Lefschetz property.*

Proof. Let $\ell \in R$ be a general linear form, and put $s = \min\{j \in \mathbb{Z} \mid [I]_j \neq 0\}$. As $[R]_i = [R/I]_i$ for $i < s$ and multiplication by ℓ on R is injective, we see that $[R/I]_{i-1} \rightarrow [R/I]_i$ is injective if $i < s$. Moreover, since $R/(I, \ell) \cong K[x]/(x^s)$ and $[K[x]/(x^s)]_i = 0$ for $i \geq s$, the map $[R/I]_{i-1} \rightarrow [R/I]_i$ has a trivial cokernel if $i \geq s$, that is, the map is surjective if $i \geq s$. Hence R/I has the weak Lefschetz property. \square

8.2. The weak Lefschetz property and perfect matchings.

If $T \subset \mathcal{T}_d$ is a triangular region, then its associated bi-adjacency matrix $Z(T)$ (see Section 5.1) admits an alternative description using multiplication by $\ell = x + y + z$.

Proposition 8.10. *Let I be a monomial ideal in $R = K[x, y, z]$, and let $\ell = x + y + z$. Fix an integer d and consider the multiplication map $\times(x + y + z) : [R/I]_{d-2} \rightarrow [R/I]_{d-1}$. Let $M(d)$ be the matrix to this linear map with respect to the monomial bases of $[R/I]_{d-2}$ and $[R/I]_{d-1}$ in reverse-lexicographic order. Then the transpose of $M(d)$ is the bi-adjacency matrix $Z(T_d(I))$.*

Proof. Set $s = h_{R/I}(d-2)$ and $t = h_{R/I}(d-1)$, and let $\{m_1, \dots, m_s\}$ and $\{n_1, \dots, n_t\}$ be the distinct monomials in $[R]_{d-2} \setminus I$ and $[R]_{d-1} \setminus I$, respectively, listed in reverse-lexicographic order. Then the matrix $M(d)$ is a $t \times s$ matrix. Its column j is the coordinate vector of $\ell m_j = xm_j + ym_j + zm_j$ modulo I with respect to the chosen basis of $[R/I]_{d-1}$. In particular, the entry in column j and row i is 1 if and only if n_i is a multiple of m_j .

Recall from Subsection 5.1 that the rows and of $Z(T_d(I))$ are indexed by the downward- and upward-pointing unit triangles, respectively. These triangles are labeled by the monomials in $[R]_{d-2} \setminus I$ and $[R]_{d-1} \setminus I$, respectively. Since the label of an upward-pointing triangle is a multiple of the label of a downward-pointing triangle if and only if the triangles are adjacent, it follows that $Z(T_d(I)) = M(d)^T$. \square

For ease of reference, we record the following consequence.

Corollary 8.11. *Let I be a monomial ideal in $R = K[x, y, z]$. Then the multiplication map $\times(x + y + z) : [R/I]_{d-2} \rightarrow [R/I]_{d-1}$ has maximal rank if and only if the matrix $Z(T_d(I))$ has maximal rank.*

Combined with Proposition 8.6, we get a criterion for the presence of the weak Lefschetz property.

Corollary 8.12. *Let I be an Artinian monomial ideal in $R = K[x, y, z]$. Then R/I has the weak Lefschetz property if and only if, for each positive integer d , the matrix $Z(T_d(I))$ has maximal rank.*

Assuming large enough socle degrees, it is enough to consider at most two explicit matrices to check for the weak Lefschetz property.

Corollary 8.13. *Let I be an Artinian monomial ideal in $R = K[x, y, z]$, and suppose the degrees of the socle generators of R/I are at least $d - 2$. Then:*

- (i) *If $0 \neq h_{R/I}(d - 1) = h_{R/I}(d)$, then R/I has the weak Lefschetz property if and only if $\det Z(T_d(I))$ is not zero in K .*
- (ii) *If $h_{R/I}(d - 2) < h_{R/I}(d - 1)$ and $h_{R/I}(d - 1) > h_{R/I}(d)$, then R/I has the weak Lefschetz property if and only if $Z(T_d(I))$ and $Z(T_{d+1}(I))$ both have a maximal minor that is not zero in K .*

Proof. By Proposition 8.6, it is enough to check whether $\ell = x + y + z$ is a Lefschetz element of R/I . Hence, the result follows by combining Corollary 8.11 and Proposition 8.4 and Corollary 8.5, respectively. \square

In the case, where the region $T_d(I)$ is balanced, we interpreted the determinant of $Z(T_d(I))$ as the signed enumeration of perfect matchings on the bipartite graph $G(T)$ (see Subsection 5.1). In general, we can similarly interpret the maximal minors of $Z(T_d(I))$ by removing unit triangles from $T_d(I)$, since the rows and columns of $Z(T_d(I))$ are indexed by the triangles of $T_d(I)$. More precisely, let $T = T_d(I)$ be a ∇ -heavy triangular region with k more downward-pointing triangles than upward-pointing triangles. Abusing notation slightly, we define a *maximal minor* of T to be a balanced subregion U of T that is obtained by removing k downward-pointing triangles from T . Similarly, if T is \triangle -heavy, then we remove only upward-pointing triangles to get a maximal minor.

Clearly, if U is a maximal minor of T , then $\det Z(U)$ is indeed a maximal minor of $Z(T)$. Thus, $Z(T)$ has maximal rank if and only if there is a maximal minor U of T such that $Z(U)$ has maximal rank.

Example 8.14. Let $I = (x^4, y^4, z^4, x^2z^2)$. Then the Hilbert function of R/I , evaluated between degrees 0 and 7, is $(1, 3, 6, 10, 11, 9, 6, 2)$, and R/I is level with socle degree 7. Hence, by Corollary 8.13, R/I has the weak Lefschetz property if and only if $Z(T_5(I))$ and $Z(T_6(I))$ both have a maximal minor of maximal rank.

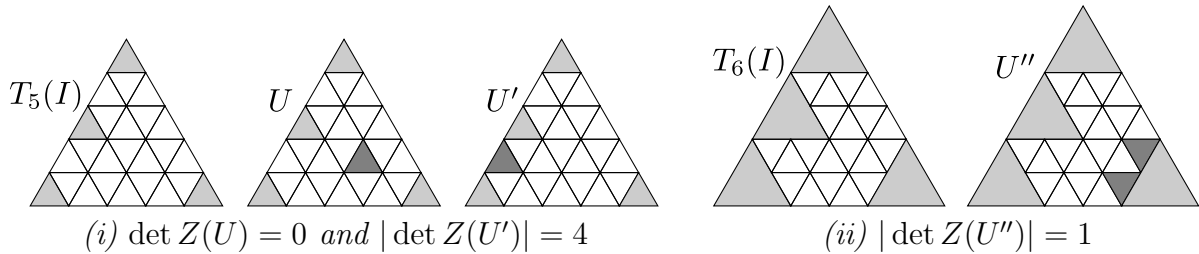


FIGURE 8.1. Examples of maximal minors of $T_d(I)$, where $I = (x^4, y^4, z^4, x^2z^2)$.

Since $h_{R/I}(3) = 10 < h_{R/I}(4) = 11$, we need to remove 1 upward-pointing triangle from $T_5(I)$ to get a maximal minor of $T_5(I)$; see Figure 8.1(i) for a pair of examples. There are $\binom{11}{10} = 11$ maximal minors, and these have signed enumerations with magnitudes 0, 4, and 8. Thus multiplication from degree 3 to degree 4 fails injectivity exactly if the characteristic of K is 2.

Furthermore, since $h_{R/I}(4) = 11 > h_{R/I}(5) = 9$, we need to remove 2 downward-pointing triangles from $T_6(I)$ to get a maximal minor of $T_6(I)$; see Figure 8.1(ii) for an example.

There are $\binom{11}{9} = 55$ maximal minors, and these have signed enumerations with magnitudes 0, 1, 2, 3, 5, and 8. Thus multiplication from degree 4 to degree 5 is always surjective (choose the maximal minor whose signed enumeration is 1).

Hence, we conclude that R/I has the weak Lefschetz property if and only if the characteristic of the base field is *not* 2.

8.3. The weak Lefschetz property and lattice paths.

The key observation in this section is that the lattice path matrix $N(T_d(I))$ (see Section 5.2) can be used to study the cokernel of multiplication by $\ell = x + y + z$ on R/I .

Let I be a monomial ideal of $R = K[x, y, z]$. Then the cokernel of the multiplication map $\times(x + y + z) : [R/I]_{d-2} \rightarrow [R/I]_{d-1}$ is $[R/(I, x + y + z)]_{d-1}$. This is isomorphic to $[S/J]_{d-1}$, where $S = K[x, y]$ and J is the ideal generated by the generators of I with $x + y$ substituted for z .

Proposition 8.15. *Let I be a monomial ideal of $R = K[x, y, z]$, and let J be the ideal of $S = K[x, y]$ generated by the generators of I with $x + y$ substituted for z . Fix an integer d and set $N = N(T_d(I))$. Then $\dim_K [S/J]_{d-1} = \dim_K \ker N^T$.*

Proof. First, we describe a matrix whose rank equals $\dim_K [J]_{d-1}$. Define an integer a as the least power of x in I that is less than d , and set $a := d$ if no such power exist. Similarly, define an integer $b \leq d$ using powers of y in I . Let G_1 and G_3 be the sets of monomials in $x^a[S]_{d-1-a}$ and $y^b[S]_{d-1-b}$, respectively. Furthermore, let G_2 be the set consisting of the polynomials $x^p y^{d-1-p-e} (x + y)^e \in [J]_{d-1}$ such that $x^i y^j z^e$ is a minimal generator of I , where $e > 0$, $i \leq p$, and $j \leq d - 1 - p - e$. Replacing $x + y$ by z , each element of G_2 corresponds to a monomial $x^p y^{d-1-p-e} z^e \in [I]_{d-1}$. Order the elements of G_2 by using the reverse-lexicographic order of the corresponding monomials in $[I]_{d-1}$, from smallest to largest. Similarly, order the monomials in G_1 and G_3 reverse-lexicographically, from smallest to largest. Note that $G_1 \cup G_2 \cup G_3$ is a generating set for the vector space $[J]_{d-1}$. The coordinate vector of a polynomial in $[S]_{d-1}$ with respect to the monomial basis of $[S]_{d-1}$ has as entries the coefficients of the monomials in $[S]_{d-1}$. Order this basis again reverse-lexicographically from smallest to largest. Now let M be the matrix whose column vectors are the coordinate vectors of the polynomials in G_1, G_2 , and G_3 , listed in this order. Then $\dim_K [J]_{d-1} = \text{rank } M$ because G generates $[J]_{d-1}$.

Second, consider the lattice path matrix $N = N(T_d(I))$. Its rows and columns are indexed by the starting and end points of lattice paths, respectively. Fix a starting point A_i and an end point E_j . The monomial label of A_i is of the form $x^s y^{d-1-s}$, where $x^s y^{d-1-s} \notin I$. Thus, the orthogonalised coordinates of $A_i \in \mathbb{Z}^2$ are (s, s) . The monomial label of the end point E_j is of the form $x^p y^{d-1-p-e} z^e$, where $x^p y^{d-1-p-e} z^e$ is a multiple of a minimal generator of I of the form $x^i y^j z^e$ with the same exponent of z . The orthogonalised coordinates of E_j are $(p + e, p)$. Hence there are

$$\begin{pmatrix} p + e - s + s - p \\ s - p \end{pmatrix} = \begin{pmatrix} e \\ s - p \end{pmatrix}$$

lattice paths in \mathbb{Z}^2 from A_i to E_j . By definition, this is the (i, j) -entry of the lattice path matrix N .

The monomial label of the end point E_j corresponds to the polynomial

$$x^p y^{d-1-p-e} (x + y)^e = \sum_{k=0}^e \binom{e}{k} x^{p+k} y^{d-1-p-k}$$

in G_2 . Thus, its coefficient of the monomial label $x^s y^{d-1-s}$ is $N_{(i,j)}$. It follows that the matrix M has the form

$$M = \begin{pmatrix} \mathcal{I}_{d-b} & * & 0 \\ 0 & N & 0 \\ 0 & * & \mathcal{I}_{d-a} \end{pmatrix},$$

where we used \mathcal{I}_k to denote the $k \times k$ identity matrix.

Notice that the matrices M and N have $d = \dim_K[S]_{d-1}$ and $a + b - d$ rows, respectively. We conclude that

$$\begin{aligned} \dim_K[S/J]_{d-1} &= d - \dim_K[J]_{d-1} \\ &= d - \text{rank } M \\ &= a + b - d - \text{rank } N \\ &= \dim_K \ker N^T, \end{aligned}$$

as claimed. \square

The last result provides another way for checking whether the multiplication by $x + y + z$ has maximal rank.

Corollary 8.16. *Let I be a monomial ideal in $R = K[x, y, z]$. Then the multiplication map $\varphi_d = \times(x + y + z) : [R/I]_{d-2} \rightarrow [R/I]_{d-1}$ has maximal rank if and only if $N = N(T_d(I))$ has maximal rank.*

Proof. Consider the exact sequence

$$[R/I]_{d-2} \xrightarrow{\varphi_d} [R/I]_{d-1} \longrightarrow [S/J]_{d-1} \longrightarrow 0.$$

It gives that φ_d has maximal rank if and only if $\dim_K[S/J]_{d-1} = \max\{0, \dim_K[R/I]_{d-1} - \dim_K[R/I]_{d-2}\}$. By Proposition 8.15, this is equivalent to

$$\dim_K \ker N^T = \max\{0, \dim_K[R/I]_{d-1} - \dim_K[R/I]_{d-2}\}.$$

Recall that, by construction, the vertices of the lattice $L(T_d(I))$ are on edges of the triangles that are parallel to the upper-left edge of \mathcal{T}_d , where this edge belongs to just an upward-pointing triangle (A -vertices), just a downward-pointing triangle (E -vertices), or an upward- and a downward-pointing unit triangle (all other vertices). Suppose there are m A -vertices, n E -vertices, and t other vertices. Then there are $m + t$ upward-pointing triangles and $n + t$ downward-pointing triangles, that is, $\dim_K[R/I]_{d-1} = m + t$ and $\dim_K[R/I]_{d-2} = n + t$. Hence

$$\dim_K[R/I]_{d-1} - \dim_K[R/I]_{d-2} = (m + t) - (n + t) = m - n.$$

Since the rows and columns of N are indexed by A - and E -vertices, respectively, N is an $m \times n$ matrix. Hence, N has maximal rank if and only if

$$\dim_K \ker N^T = \max\{0, m - n\} = \max\{0, \dim_K[R/I]_{d-1} - \dim_K[R/I]_{d-2}\}.$$

\square

The last argument shows in particular that, for any region $T \subset \mathcal{T}_d$, the bi-adjacency matrix $Z(T)$ is a square matrix if and only if the lattice path matrix $N(T)$ is a square matrix. Hence, using Corollary 8.16 instead of Corollary 8.11, we obtain a result that is analogous to Corollary 8.13.

Corollary 8.17. *Let I be an Artinian monomial ideal in $R = K[x, y, z]$, and suppose the degrees of the socle generators of R/I are at least $d - 2$. Then:*

- (i) *If $0 \neq h_{R/I}(d - 1) = h_{R/I}(d)$, then R/I has the weak Lefschetz property if and only if $\det N(T_d(I))$ is not zero in K .*
- (ii) *If $h_{R/I}(d - 2) < h_{R/I}(d - 1)$ and $h_{R/I}(d - 1) > h_{R/I}(d)$, then R/I has the weak Lefschetz property if and only if $N(T_d(I))$ and $N(T_{d+1}(I))$ both have a maximal minor that is not zero in K .*

In the case where $T = T_d(I)$ is balanced we interpreted the determinant of $N(T)$ as the signed enumeration of families of non-intersecting lattice paths in the lattice $L(T)$ (see Subsection 5.2). In general, we can similarly interpret the maximal minors of $N(T)$ by removing A -vertices or E -vertices from $L(T)$, since the rows and columns of $N(T)$ are indexed by these vertices. Note that removing the A - and E -vertices is the same as removing the associated unit triangles in T . For example, U' in Figure 8.1(i) corresponds to removing the starting point A_1 from U . It follows that the maximal minors of $N(T)$ are exactly the determinants of maximal minors of T that are obtained from T by removing only unit triangles corresponding to A - and E -vertices. We call such a maximal minor a *restricted maximal minor* of T .

Clearly, $N(T)$ has maximal rank if and only if there is a restricted maximal minor U of T such that $N(U)$ has maximal rank. As a consequence, for a \triangle -heavy region T , it is enough to check the restricted maximal minors in order to determine whether $Z(T)$ has maximal rank.

Proposition 8.18. *Let $T = T_d(I)$ be an \triangle -heavy triangular region. Then $Z(T)$ has maximal rank if and only if there is a restricted maximal minor U of T such that $Z(U)$ has maximal rank.*

Proof. By Corollaries 8.11 and 8.16, we have that $Z(T)$ has maximal rank if and only if $N(T)$ has maximal rank. Since each restricted maximal minor U of T is obtained by removing upward-pointing triangles, it is the triangular region of some monomial ideal. Thus, Theorem 5.17 gives $|\det Z(U)| = |\det N(U)|$. \square

Remark 8.19. The preceding proposition allows us to reduce the number of minors of $Z(T)$ that need to be considered. In Example 8.14(i), there are 11 maximal minors of $T_5(I)$, but only 2 *restricted* maximal minors.

In the special case of triangular regions as in Proposition 6.5, Proposition 8.18 was observed by Li and Zanello in [38, Theorem 3.2].

We continue to consider Example 8.14, using lattice path matrices now.

Example 8.20. Recall the ideal $I = (x^4, y^4, z^4, x^2z^2)$ from Example 8.14. By Corollary 8.17, R/I has the weak Lefschetz property if and only if $N(T_5(I))$ and $N(T_6(I))$ have maximal rank. Since $N(T_5(I))$ is a 2×1 matrix, we need to remove 1 A -vertex to get a maximal minor (see U' in Figure 8.1(i) for one of the two choices). Both choices have signed enumeration 4. Since $N(T_6(I))$ is a 0×2 matrix we need to remove 2 E -vertices to get a restricted maximal minor. The region U'' in Figure 8.1(ii) is the only choice, and the signed enumeration is 1. Thus, we see again that R/I has the weak Lefschetz property if and only if the base field K has not characteristic 2.

8.4. Complete Intersections.

We now begin to reinterpret the results in Section 6 about signed enumerations of triangular regions as results about the weak Lefschetz property for the associated Artinian ideals. In this subsection we restrict ourselves to the ideals with the fewest number of generators, namely the ideals of the form $I = (a^a, y^b, z^c)$. These are monomial complete intersections, and the question whether they have the weak Lefschetz property has motivated a great deal of research (see [47] and Remark 8.23 below).

We recall a well-known result of Reid, Roberts, and Roitman about the shape of Hilbert functions of monomial complete intersections.

Lemma 8.21. [54, Theorem 1] *Let $I = (a^a, y^b, z^c)$, where a, b , and c are positive integers. Then the Hilbert function $h = h_{R/I}$ of R/I has the following properties:*

- (i) $h(j-2) < h(j-1)$ if and only if $1 \leq j < \min\{a+b, a+c, b+c, \frac{1}{2}(a+b+c)\}$;
- (ii) $h(j-2) = h(j-1)$ if and only if $\min\{a+b, a+c, b+c, \frac{1}{2}(a+b+c)\} \leq j \leq \max\{a, b, c, \frac{1}{2}(a+b+c)\}$; and
- (iii) $h(j-2) > h(j-1)$ if and only if $\max\{a, b, c, \frac{1}{2}(a+b+c)\} < j \leq a+b+c-1$.

Depending on the characteristic of the base field we get the following sufficient conditions that guarantee the weak Lefschetz property.

Proposition 8.22. *Let $I = (x^a, y^b, z^c)$, where a, b , and c are positive integers. Set $d = \lfloor \frac{a+b+c}{2} \rfloor$. Then:*

- (i) *If $d < \max\{a, b, c\}$, then R/I has the weak Lefschetz property, regardless of the characteristic of K .*
- (ii) *If $a+b+c$ is even, then R/I has the weak Lefschetz property in characteristic p if and only if p does not divide $\text{Mac}(d-a, d-b, d-c)$.*
- (iii) *If $a+b+c$ is odd, then R/I has the weak Lefschetz property in characteristic p if and only if p does not divide at least one of the integers*

$$\frac{\binom{d-1}{a-1}}{\binom{d-1}{i}} \binom{d-c}{a-i-1} \text{Mac}(d-a-1, d-b, d-c),$$

where $d-1-b < i < a$.

In any case, R/I has the weak Lefschetz property in characteristic p if $p = 0$ or $p \geq \lfloor \frac{a+b+c}{2} \rfloor$.

Proof. The algebra R/I has exactly one socle generator. It has degree $a+b+c-3 \geq d-2$.

If $d < \max\{a, b, c\}$, then without loss of generality we may assume $a > d$, that is, $a > b+c$. In this case, $T_d(I)$ has two punctures, one of length $d-b$ and one of length $d-c$. Moreover, $d-b+d-c = a > d$ so the two punctures overlap. Hence $T_d(I)$ is balanced and has a unique tiling. That is, $|\det Z(T)| = 1$ and so R/I has the weak Lefschetz property, regardless of the characteristic of K (see Corollary 8.13).

Suppose $d \geq \max\{a, b, c\}$. By Lemma 8.21, we have $h_{R/I}(d-2) \leq h_{R/I}(d-1) > h_{R/I}(d)$.

Assume $a+b+c$ is even. Then Proposition 6.5 gives that $|\det Z(T_d(I))| = \text{Mac}(d-a, d-b, d-c)$, and so claim (ii) follows by Corollary 8.13.

Assume $a+b+c$ is odd, and so $d = \frac{1}{2}(a+b+c-1)$. In this case it is enough to find non-trivial maximal minors of $T_d(I)$ and $T_{d+1}(I)$ by Corollary 8.13. Consider the hexagonal regions formed by the present unit triangles of each $T_{d+1}(I)$ and $T_d(I)$. The former hexagon is obtained from the latter by a rotation about 180° . Thus, we need only consider the

maximal minors of $T_d(I)$. This region has exactly one more upward-pointing triangle than downward-pointing triangle. Hence, by Proposition 8.18, it suffices to check whether the restricted maximal minors of $T_d(I)$ have maximal rank. These minors are exactly $T_i := T_d(x^a, y^b, z^c, x^i y^{d-1-i})$, where $d-1-b < i < a$. Using Proposition 6.11, we get that $|\det Z(T_i)|$ is

$$\text{Mac}(a-1-i, d-a, 1) \text{Mac}(i+b-d, d-b, 1) \frac{\mathcal{H}(d-a+1)\mathcal{H}(d-b+1)\mathcal{H}(d-c+1)\mathcal{H}(d)}{\mathcal{H}(a)\mathcal{H}(b)\mathcal{H}(c)\mathcal{H}(1)},$$

where we notice $d - (i + (d-1-i)) = 1$. Since $\text{Mac}(n, k, 1) = \binom{n+k}{k}$ and $\mathcal{H}(n) = (n-1)!\mathcal{H}(n-1)$, for positive integers n and k , we can rewrite $|\det Z(T_i)|$ as

$$\binom{d-1-i}{d-a} \binom{i}{d-b} \frac{(d-b)!(d-c)!}{(a-1)!} \text{Mac}(d-a-1, d-b, d-c).$$

Simplifying this expression, we get part (iii).

Finally, using both Propositions 6.5 and 6.11 we see that the prime divisors of $|\det Z(T_i)|$ are bounded above by $d-1$ in each case. \square

As announced, we briefly comment on the history of the last result and the research it motivated.

Remark 8.23. The presence of the weak Lefschetz property for monomial complete intersections has been studied by many authors. The fact that *all* monomial complete intersections, in any number of variables, have the strong Lefschetz property in characteristic zero was proven first by Stanley [56] using the Hard Lefschetz Theorem. (See [14], and the references contained therein, for more on the history of this theorem.) However, the weak Lefschetz property can fail in positive characteristic.

The weak Lefschetz property in arbitrary characteristic in the case where one generator has much larger degree than the others (case (i) in the preceding proposition) was first established by Watanabe [62, Corollary 2] for arbitrary complete intersections in three variables, not just monomial ones. Migliore and Miró-Roig [44, Proposition 5.2] generalised this to complete intersections in n variables.

Part (ii) of the above result was first established by the authors [15, Theorem 4.3] (with an extra generator of sufficiently large degree), and independently by Li and Zanello [38, Theorem 3.2]. The latter also proved part (iii) above (use $i = a - k$). However, while both papers mentioned the connection to lozenge tilings of hexagons, it was Chen, Guo, Jin, and Li [10] who provided the first combinatorial explanation. In particular, the case (ii) was studied in [10, Theorem 1.2]. We also note that [38, Theorem 4.3] can be recovered from Proposition 8.22 if we set $a = \beta + \gamma$, $b = \alpha + \gamma$, and $c = \alpha + \beta$.

More explicit results have been found in the special case where all generators have the same degree, i.e., $I_a = (x^a, y^a, z^a)$. Brenner and Kaid used the idea of a syzygy gap to explicitly classify the prime characteristics in which I_a has the weak Lefschetz property [9, Theorem 2.6]. Kustin, Rahmati, and Vraciu used this result in [36], in which they related the presence of the weak Lefschetz property of R/I_a to the finiteness of the projective dimension of $I_a : (x^n + y^n + z^n)$. Moreover, Kustin and Vraciu later gave an alternate explicit classification of the prime characteristics in which I_a has the weak Lefschetz property [37, Theorem 4.3].

As a final note, Kustin and Vraciu [37] also gave an explicit classification of the prime characteristics in which monomial complete intersections in arbitrarily many variables with

all generators of the same degree have the weak Lefschetz property. This was expanded by the first author [14, Theorem 7.2] to an explicit classification of the prime characteristics, in which the algebra has the *strong Lefschetz property*. In this work another combinatorial connection was used to study the presence of the weak Lefschetz property for monomial complete intersections in arbitrarily many variables.

8.5. Reinterpretation.

We provide some rather direct interpretations of earlier results to more general ideals than complete intersections. More involved uses of our methods will be described in the following sections.

Here we will restrict ourselves to considering balanced regions. In this case we observe the following necessary condition for the presence of the weak Lefschetz property.

Proposition 8.24. *Let I be a monomial ideal such that $T_d(I)$ is a balanced region that is not tileable. Put $J = I + (x^d, y^d, z^d)$. Then R/J never has the weak Lefschetz property, regardless of the characteristic of K .*

Proof. Since $T_d(I) = T_d(J)$ is not tileable, Theorem 5.5 gives $\det Z(T_d(J)) = 0$. Thus, $Z(T_d(J))$ does not have maximal rank. Now we conclude by Corollary 8.12. \square

We illustrate the preceding proposition with an example.

Example 8.25. Consider the regions depicted in Figure 8.2. These regions are both bal-

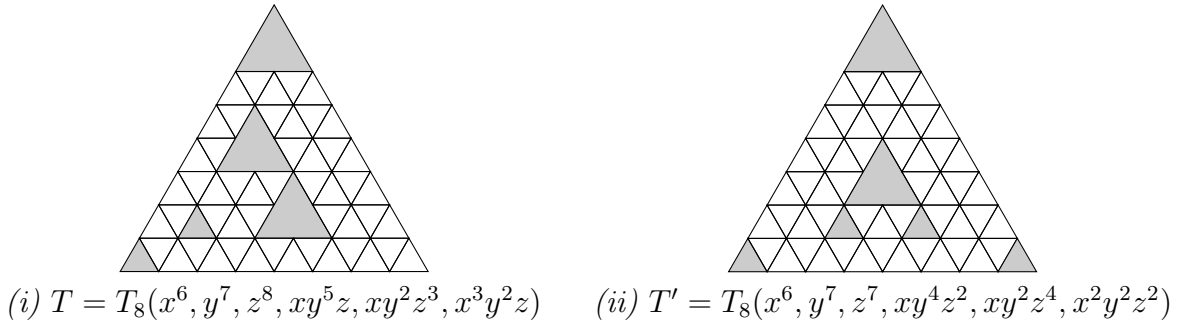


FIGURE 8.2. Two balanced non-tileable triangular regions.

anced, but non-tileable as they contain ∇ -heavy monomial subregions (see Theorem 3.2). In particular, the monomial subregion associated to xy^2z in T and the monomial subregion associated to xy^2z^2 in T' are both ∇ -heavy. Thus, $R/(x^6, y^7, z^8, xy^5z, xy^2z^3, x^3y^2z)$ and $R/(x^6, y^7, z^7, xy^4z^2, xy^2z^4, x^2y^2z^2)$ both fail to have the weak Lefschetz property, regardless of the characteristic of the base field.

Now we use Proposition 8.24 in order to relate the weak Lefschetz property and semistability of syzygy bundles (see Section 4). In preparation, we record the following observation. Recall that the monomial ideal of a triangular region $T \subset \mathcal{T}_d$ is the largest ideal J whose minimal generators have degrees less than d such that $T = T_d(J)$ (see Subsection 2.5).

Lemma 8.26. *Let $J \subset R$ be the monomial ideal of a triangular region $T \subset \mathcal{T}_d$. Then:*

- (i) *The region T has no overlapping punctures if and only if each degree of a least common multiple of two distinct minimal generators of J is at least d .*

- (ii) *The punctures of T are not overlapping nor touching if and only if each degree of a least common multiple of two distinct minimal generators of J is at least $d + 1$.*

Proof. Let m_1 and m_2 be two distinct minimal generators of J . Then their associated punctures overlap if and only if there is a monomial of degree $d - 1$ that is a multiple of m_1 and m_2 . The existence of such a monomial means precisely that the degree of the least common multiple of m_1 and m_2 is at most $d - 1$. Now claim (i) follows.

Assertion (ii) is shown similarly by observing that the punctures to m_1 and m_2 touch if and only if there is a monomial of degree d that is a multiple of m_1 and m_2 . \square

The following consequence is useful later on.

Corollary 8.27. *Assume $T \subset \mathcal{T}_d$ is a triangular region whose punctures are not overlapping nor touching, and let J be the monomial ideal of T . Then R/J does not have non-zero socle elements of degree less than $d - 1$.*

Proof. Since J is a monomial ideal, every first syzygy of J corresponds to a relation $m_i n_i - m_j n_j = 0$ for suitable monomials n_i and n_j , where m_i and m_j are distinct monomial minimal generators of J . Applying Lemma 8.26 to the equality $m_i n_i = m_j n_j$, we conclude that the degree of each first syzygy of J is at least $d + 1$. It follows that the degree of every second syzygy of J is at least $d + 2$. Each minimal second syzygy of J corresponds to a socle generator of R/J (see the beginning of Subsection 8.1). Hence, the degrees of the socle generators of R/J are at least $d - 1$. \square

The converse of Corollary 8.27 is not true in general. For example, the socle generators of $R/(x^6, y^7, z^8, xy^5z, xy^2z^3, x^3y^2z)$ have degrees greater than 7, but two punctures of $T_8(x^6, y^7, z^8, xy^5z, xy^2z^3, x^3y^2z)$ touch each other (see Figure 8.2).

Recall that perfectly-punctured regions were defined above Corollary 3.4. This concept is used in the proof of the following result.

Theorem 8.28. *Let $I \subset R$ be an Artinian ideal whose minimal monomial generators have degrees d_1, \dots, d_t . Set*

$$d := \frac{d_1 + \dots + d_t}{t - 1}.$$

Assume $\text{char } K = 0$ and that the following conditions are satisfied:

- (i) *The number d is an integer.*
- (ii) *For all $i = 1, \dots, t$, one has $d > d_i$.*
- (iii) *Each degree of a least common multiple of two distinct minimal generators of I is at least d .*

Then the syzygy bundle of I is semistable if R/I has the weak Lefschetz property.

Proof. Consider the triangular region $T = T_d(I)$. By assumption (iii) and Lemma 8.26, we obtain that the punctures of T do not overlap. Recall that the side length of the puncture to a minimal generator of degree d_i is $d - d_i$. The definition of d is equivalent to

$$d = \sum_{i=1}^t (d - d_i).$$

We conclude that the region T is balanced and perfectly-punctured. Combined with the weak Lefschetz property of R/I , the first property implies that T is tileable by Proposition 8.24. Now Theorem 4.5 gives the semistability of the syzygy bundle of I . \square

The converse of the above result is not true, in general.

Remark 8.29. The mirror symmetric regions considered in Section 7 are all balanced and tileable. Thus, Theorem 4.5 gives that each ideal of such a region (see Remark 7.3)

$$J = (x^{h_1}, y^{d-b}, z^{d-b}, x^{h_2}(yz)^{\frac{1}{2}(d-d_2-h_2)}, \dots, x^{h_m}(yz)^{\frac{1}{2}(d-d_m-h_m)}).$$

has a semi-stable syzygy bundle. However, Theorem 7.4 shows that R/J does not have the weak Lefschetz property if the number of axial punctures of $T_d(J)$ with odd side length is 2 or 3 modulo 4. If, instead, all axial punctures, except possibly the top one, do have an even side length, then R/J has the weak Lefschetz property (see Proposition 7.7).

However, under stronger assumptions the converse to Theorem 8.28 is indeed true.

Theorem 8.30. *Let $I \subset R$ be an Artinian ideal with minimal monomial generators m_1, \dots, m_t . Set*

$$d := \frac{d_1 + \dots + d_t}{t-1},$$

where $d_i = \deg m_i$. Assume $\text{char } K = 0$ and that the following conditions are satisfied:

- (i) The number d is an integer.
- (ii) For all $i = 1, \dots, t$, one has $d > d_i$.
- (iii) If $i \neq j$, then the degree of the least common multiple of m_i and m_j is at least $d+1$.
- (iv) If m_i is not a power of x, y , or z , then $d - d_i$ is even.

Then the syzygy bundle of I is semistable if and only if R/I has the weak Lefschetz property.

Proof. By Theorem 8.28, it is enough to show that R/I has the weak Lefschetz property if the syzygy bundle of I is semistable.

Consider the region $T = T_d(I)$. In the proof of Theorem 8.28 we showed that T is balanced and perfectly-punctured. Hence T is tileable by Theorem 4.5. Since all floating punctures of T have an even side length by assumption (iv), Theorem 5.5 and Proposition 6.15 give that $Z(T)$ has maximal rank.

Assumption (iii) means that the punctures of T are not overlapping nor touching (see Lemma 8.26). Hence, Corollary 8.27 yields that the degrees of the socle generators of R/I are at least $d-1$. Therefore, Corollary 8.5 proves that R/I has the weak Lefschetz property. \square

We now show that, for all positive integers d_1, \dots, d_t with $t \geq 3$ that satisfy the numerical assumptions (i), (ii), and (iv) of Theorem 8.30, there is a monomial ideal I whose minimal generators have degrees d_1, \dots, d_t to which Theorem 8.30 applies and guarantees the weak Lefschetz property of R/I .

Example 8.31. Let d_1, \dots, d_t be $t \geq 3$ positive integers satisfying the following numerical conditions:

- (i) The number $d := \frac{d_1 + \dots + d_t}{t-1}$ is an integer.
- (ii) For all $i = 1, \dots, t$, one has $d > d_i$.
- (iii) At most three of the integers $d - d_i$ are not even.

Re-indexing if needed, we may assume that $d_3 \leq \min\{d_1, d_2\}$ and that $d - d_i$ is even whenever $4 \leq i \leq t$. Consider the following ideal

$$I = (x^{d_1}, y^{d_2}, z^{d_3}, m_4, \dots, m_t),$$

where $m_4 = x^{d-d_3}yz^{-d-1+d_3+d_4}$ if $t \geq 4$, $m_5 = x^{2d-d_3-d_4}y^2z^{2d-2+d_3+d_4+d_5}$ if $t \geq 5$, and

$$m_i = \begin{cases} x^{d-d_3}y^{1+\sum_{k=4}^{i-1}(d-d_k)}z^{-d(i-3)-1+\sum_{k=3}^i d_k} & \text{if } 6 \leq i \leq t \text{ and } i \text{ is even} \\ x^{-1+\sum_{k=3}^{i-1}(d-d_k)}y^2z^{-d(i-3)-1+\sum_{k=3}^i d_k} & \text{if } 7 \leq i \leq t \text{ and } i \text{ is odd.} \end{cases}$$

Note that $\deg m_i = d_i$ for all i . One easily checks that the degree of the least common multiple of any two distinct minimal generators of I is at least $d+1$, that is, the punctures of $T_d(I)$ do not overlap nor touch each other.

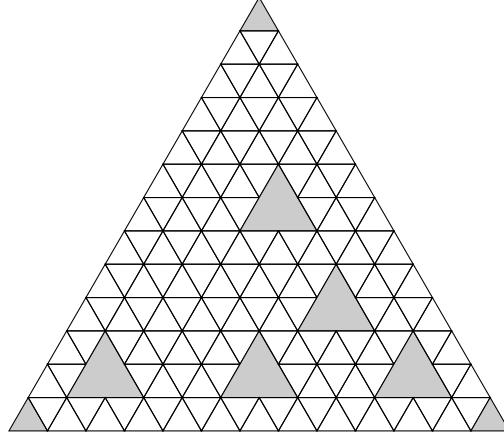


FIGURE 8.3. The region corresponding to $d_1 = d_2 = d_3 = 12$ and $d_4 = \dots = d_8 = 11$ in Example 8.31.

Corollary 8.32. *Let I be any ideal as defined in Example 8.31. Assume that the base field K has characteristic zero. Then R/I has the weak Lefschetz property and the syzygy bundle of I is semistable.*

Proof. By construction, the considered ideals satisfy assumptions (i)–(iv) of Theorem 8.30. Furthermore, the region $T_d(I)$ has no over-punctured monomial subregions. Hence, it is tileable by Corollary 3.4. (Alternatively, one can exhibit a family of non-intersecting lattice paths to check tileability.) By Theorem 4.5, it follows that the syzygy bundle of I is semistable, and hence R/I has the weak Lefschetz property by Theorem 8.30. \square

Remark 8.33. Given an integer $t \geq 3$, there are many choices for the integers d_1, \dots, d_t , and thus for the ideals exhibited in Example 8.31. A convenient choice, for which the description of the ideal becomes simpler, is $d_1 = 2t - 4$, $d_2 = d_3 = d - 1$, and $d_4 = \dots = d_t = d - 2$, where d is any integer satisfying $d \geq 2t - 3$. Then the corresponding ideal is

$$I = (x^{2t-4}, y^{d-1}, z^{d-1}, xyz^{d-4}, x^3y^2z^{d-7}, m_6, \dots, m_t),$$

where

$$m_i = \begin{cases} xy^{2i-7}z^{d+4-2i} & \text{if } 6 \leq i \leq t \text{ and } i \text{ is even} \\ x^{2i-8}y^2z^{d+4-2i} & \text{if } 7 \leq i \leq t \text{ and } i \text{ is odd.} \end{cases}$$

The ideal in Figure 8.3 is generated as above with $d = 13$ and $t = 8$.

9. ARTINIAN MONOMIAL ALGEBRAS OF TYPE TWO IN THREE VARIABLES

Boij, Migliore, Miró-Roig, Zanello, and the second author proved in [3, Theorem 6.2] that the Artinian monomial algebras of type two in three variables that are *level* have the weak Lefschetz property in characteristic zero. The proof given there is surprisingly intricate and lengthy. In this section, we establish a more general result using techniques derived in the previous sections.

To begin, we classify the Artinian monomial ideals I in $R = K[x, y, z]$ such that R/I has type two, that is, its socle is of the form $\text{soc}(R/I) \cong K(-s) \oplus K(-t)$. The algebra R/I is level if the socle degrees s and t are equal. The classification in the level case has been established in [3, Proposition 6.1]. The following more general result is obtained similarly.

Lemma 9.1. *Let I be an Artinian monomial ideal in $R = K[x, y, z]$ such that R/I is of type 2. Then, up to a change of variables, I has one of the following two forms:*

- (i) $I = (x^a, y^b, z^c, x^\alpha y^\beta)$, where $0 < \alpha < a$ and $0 < \beta < b$. In this case, the socle degrees of R/I are $a + \beta + c - 3$ and $\alpha + b + c - 3$. Thus, I is level if and only if $a - \alpha = b - \beta$.
- (ii) $I = (x^a, y^b, z^c, x^\alpha y^\beta, x^\alpha z^\gamma)$, where $0 < \alpha < a$, $0 < \beta < b$, and $0 < \gamma < c$. In this case, the socle degrees of R/I are $a + \beta + \gamma - 3$ and $\alpha + b + c - 3$. Thus, I is level if and only if $a - \alpha = b - \beta + c - \gamma$.

Proof. We use Macaulay-Matlis duality. An Artinian monomial algebra of type two over R arises as the inverse system of two monomials, say $x^{a_1}y^{b_1}z^{c_1}$ and $x^{a_2}y^{b_2}z^{c_2}$, such that one does not divide the other. Thus we may assume without loss of generality that $a_1 > a_2$ and $b_1 < b_2$. We consider two cases: $c_1 = c_2$ and $c_1 \neq c_2$.

Suppose first that $c_1 = c_2$. Then the annihilator of the monomials is the ideal

$$(x^{a_1+1}, y^{b_1+1}, z^{c_1+1}) \cap (x^{a_2+1}, y^{b_2+1}, z^{c_1+1}) = (x^{a_1+1}, y^{b_2+1}, z^{c_1+1}, x^{a_2+1}y^{b_1+1}),$$

which is the form in (i). By construction, the socle elements are $x^{a_1}y^{b_1}z^{c_1}$ and $x^{a_2}y^{b_2}z^{c_1}$.

Now suppose $c_1 \neq c_2$; without loss of generality we may assume $c_1 < c_2$. Then the annihilator of the monomials is the ideal

$$(x^{a_1+1}, y^{b_1+1}, z^{c_1+1}) \cap (x^{a_2+1}, y^{b_2+1}, z^{c_2+1}) = (x^{a_1+1}, y^{b_2+1}, z^{c_2+1}, x^{a_2+1}y^{b_1+1}, x^{a_2+1}z^{c_1+1}),$$

which is the form in (ii). By construction, the socle elements are $x^{a_1}y^{b_1}z^{c_1}$ and $x^{a_2}y^{b_2}z^{c_2}$. \square

We give a complete classification of the type two algebras that have the weak Lefschetz property in characteristic zero.

Theorem 9.2. *Let I be an Artinian monomial ideal in $R = K[x, y, z]$, where K is a field of characteristic zero, such that R/I is of type 2. Then R/I fails to have the weak Lefschetz property in characteristic zero if and only if $I = (x^a, y^b, z^c, x^\alpha y^\beta, x^\alpha z^\gamma)$, up to a change of variables, where $0 < \alpha < a$, $0 < \beta < b$, and $0 < \gamma < c$, and there exists an integer d with*

$$(9.1) \quad \max \left\{ a, \alpha + \beta, \alpha + \gamma, \frac{a + \alpha + \beta + \gamma}{2} \right\} < d \\ < \min \left\{ a + \beta + \gamma, \frac{\alpha + b + c}{2}, b + c, \alpha + c, \alpha + b \right\}.$$

Proof. According to Corollary 8.12, for each integer $d > 0$, we have to decide whether the bi-adjacency matrix $Z(T_d(I))$ has maximal rank. This is always true if $d = 1$. Let $d \geq 2$.

By Lemma 9.1, we may assume that I has one of two forms given there. The difference between the two forms is an extra generator, $x^\alpha z^\gamma$. In order to determine the rank of $Z(T_d(I))$ we split $T = T_d(I)$ across the horizontal line α units from the bottom edge. We call the monomial subregion above the line, which is the subregion associated to x^α , the *upper portion* of T , denoted by T^u , and we call the isosceles trapezoid below the line the *lower portion* of T , denoted by T^l . Note that T^u is empty if $d \leq \alpha$. Both portions, T^u and T^l , are hexagons, i.e., triangular regions associated to complete intersections. In particular, if I has four generators, then $T^u = T_{d-\alpha}(x^{a-\alpha}, y^\beta, z^c)$. Similarly, if I has five generators, then $T^u = T_{d-\alpha}(x^{a-\alpha}, y^\beta, z^\gamma)$. In both cases T^l is $T_d(x^\alpha, y^b, z^c)$. See Figure 9.1 for an illustration of this decomposition.

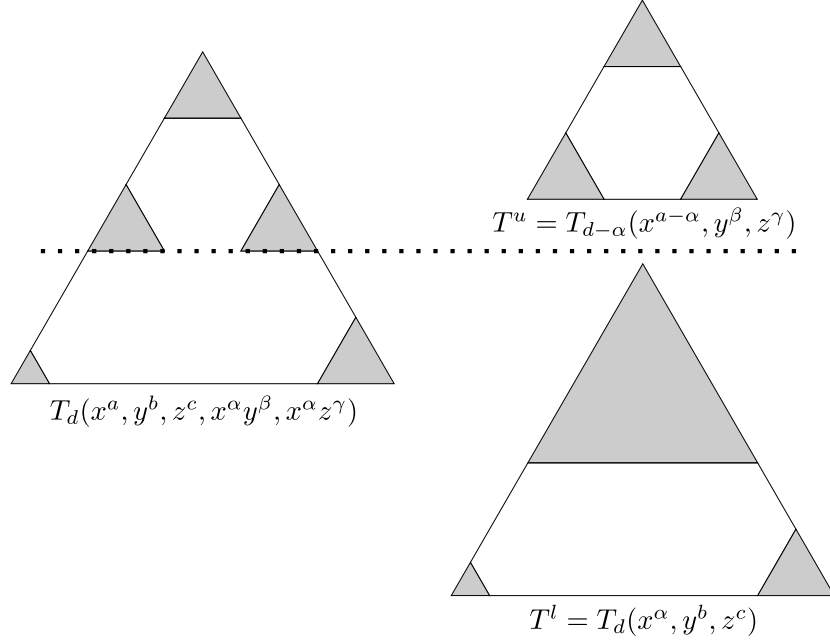


FIGURE 9.1. The decomposition of $T_d(I)$ into T^u and T^l .

After reordering rows and columns of the bi-adjacency matrix $Z(T)$, it becomes a block matrix of the form

$$(9.2) \quad Z = \begin{pmatrix} Z(T^u) & 0 \\ Y & Z(T^l) \end{pmatrix}$$

because the downward-pointing triangles in T^u are not adjacent to any upward-pointing triangles in T^l . For determining when Z has maximal rank, we study several cases, depending on whether T^u and T^l are \triangle -heavy, balanced, or ∇ -heavy.

First, suppose one of the following conditions is satisfied: (i) T^u or T^l is balanced, (ii) T^u and T^l are both \triangle -heavy, or (iii) T^u and T^l are both ∇ -heavy. In other words, T^u and T^l do not “favor” triangles of opposite orientations. Since T^u and T^l are triangular regions associated to complete intersections, both $Z(T^u)$ and $Z(T^l)$ have maximal rank by Proposition 8.22. Combining non-vanishing maximal minors of $Z(T^u)$ and $Z(T^l)$, it follows that the matrix Z has maximal rank as well.

Second, suppose T^u is \triangle -heavy and T^l is ∇ -heavy. We will show that Z has maximal rank in this case.

Let $t_u = \#\triangle(T^u) - \#\nabla(T^u)$ and $t_l = \#\nabla(T^l) - \#\triangle(T^l)$ be the number of excess triangles of each region. In a first step, we show that we may assume $t_u = t_l$. To this end we remove enough of the appropriately oriented triangles from the more unbalanced of T^u and T^l until both regions are equally unbalanced. Set $t = \min\{t_u, t_l\}$.

Assume T^u is more unbalanced, i.e., $t_u > t$. Since T^u is \triangle -heavy, the top t_u rows of \mathcal{T}_d below the puncture associated to x^a do not have a puncture. Thus, we can remove the top $t_u - t$ upward-pointing triangles in T^u along the upper-left edge of \mathcal{T}_d , starting at the puncture associated to x^a , if present, or in the top corner otherwise. Denote the resulting subregion of T by T' . Notice that Z has maximal rank if $Z(T')$ has maximal rank. Furthermore, the $t_u - t$ rows in which T and T' differ are uniquely tileable. Denote this subregion of T' by U (see Figure 9.2(i) for an illustration). By construction, the upper and the lower portion $T^{u'}$ and $T^{l'} = T^l$, respectively, of $T' \setminus U$ are equally unbalanced. Moreover, $Z(T')$ has maximal rank if and only if $Z(T' \setminus U)$ has maximal rank by Proposition 6.2. As desired, T and $T' \setminus U$ have the same shape.

Assume now that T^l is more unbalanced, i.e., $t_l > t$. Since T^l is ∇ -heavy, the two punctures associated to x^b and x^c , respectively, cover part of the bottom t_l rows of \mathcal{T}_d . Thus, we can remove the bottom $t_l - t$ downward-pointing triangles of T^l along the puncture associated to x^c . Denote the resulting subregion of T by T' . Notice that Z has maximal rank if $Z(T')$ has maximal rank. Again, the $t_l - t$ rows in which T and T' differ form a uniquely tileable subregion. Denote it by U . By construction, the upper and the lower portion $T^{u'} = T^u$ and $T^{l'}$, respectively, of $T' \setminus U$ are equally unbalanced. Moreover, $Z(T')$ has maximal rank if and only if $Z(T' \setminus U)$ has maximal rank by Proposition 6.2. As before, T and $T' \setminus U$ have the same shape.

The above discussion shows it is enough to prove that the matrix Z has maximal rank if $t_u = t_l = t$, i.e., T is balanced. Since T has no floating punctures, Proposition 6.15 gives the desired maximal rank of Z once we know that T has a tiling. To see that T' is tileable, we first place t lozenges across the line separating T^u from T^l , starting with the left-most such lozenge. Indeed, this is possible since T^u has t more upwards-pointing than downwards-pointing triangles. Next, place all fixed lozenges. The portion of T^u that remains untiled after placing these lozenges is a hexagon. Hence it is tileable. Consider now the portion of T^l that remains untiled after placing these lozenges. Since t is at most the number of horizontal rows of T^l this portion is, after a 60° rotation, a region as described in Proposition 6.11. Thus, it is tileable. Figure 9.2(ii) illustrates this procedure with an example.

It follows that T is tileable. Therefore Z has maximal rank, as desired.

Finally, suppose T^u is ∇ -heavy and T^l is \triangle -heavy. Consider any maximal minor of $Z(T)$. It corresponds to a balanced subregion T' of T . Then its upper portion $T^{u'}$ is still ∇ -heavy, and its lower portion $T^{l'}$ is \triangle -heavy. Hence, any covering of $T^{u'}$ by lozenges must also cover some upward-pointing triangles of $T^{l'}$. The remaining part of $T^{l'}$ is even more unbalanced than $T^{l'}$. This shows that T' is not tileable. Thus, $\det Z(T') = 0$ by Theorem 5.5. It follows that Z does not have maximal rank in this case.

The above case analysis proves that R/I fails the weak Lefschetz property if and only if there is an integer d so that the associated regions T^u and T^l are ∇ -heavy and \triangle -heavy, respectively. It remains to determine when this happens.

If I has only four generators, then no row of T^u has more downward-pointing than upward-pointing triangles. Hence, T^u is not ∇ -heavy. It follows that I must have five generators if R/I fails to have the weak Lefschetz property. For such an ideal I , the region $T^u =$

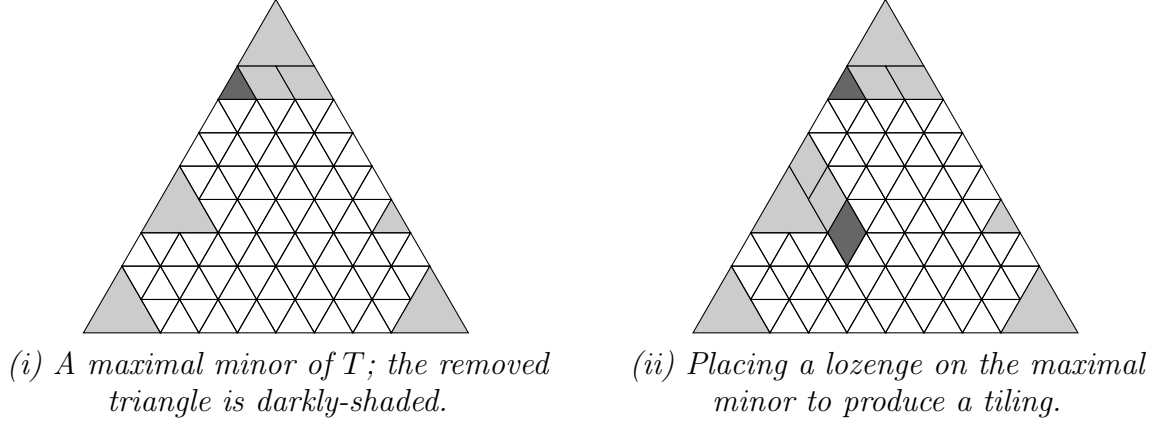


FIGURE 9.2. Let $T = T_{10}(x^8, y^8, z^8, x^3y^5, x^3z^6)$. The lightly shaded lozenges are fixed lozenges.

$T_{d-\alpha}(x^{a-\alpha}, y^\beta, z^\gamma)$ is ∇ -heavy if and only if

$$\dim_K[R/(x^{a-\alpha}, y^\beta, z^\gamma)]_{d-\alpha-2} > \dim_K[R/(x^{a-\alpha}, y^\beta, z^\gamma)]_{d-\alpha-1},$$

and $T^l = T_d(x^\alpha, y^b, z^c)$ is \triangle -heavy if and only if

$$\dim_K[R/(x^\alpha, y^b, z^c)]_{d-2} < \dim_K[R/(x^\alpha, y^b, z^c)]_{d-1}.$$

Using Lemma 8.21, a straight-forward computation shows that these two inequalities are both true if and only if d satisfies Condition 9.1. \square

Remark 9.3. The above argument establishes the following more precise version of Theorem 9.2:

Let R/I be a Artinian monomial algebra of type 2, where K is a field of characteristic zero, and let $\ell \in R$ be a general linear form. Then the multiplication map $\times \ell : [R/I]_{d-2} \rightarrow [R/I]_{d-1}$ does not have maximal rank if and only if $I = (x^a, y^b, z^c, x^\alpha y^\beta, x^\alpha z^\gamma)$, up to a change of variables, and d satisfies Condition 9.1.

Condition 9.1 in Theorem 9.2 is indeed non-vacuous.

Example 9.4. We provide three examples, the latter two come from [3, Example 6.10], with various shapes of Hilbert functions.

- (i) Let $I = (x^4, y^4, z^4, x^3y, x^3z)$. Then $d = 5$ satisfies Condition 9.1. Moreover, $T_d(I)$ is a balanced region, and R/I has a strictly unimodal Hilbert function, $(1, 3, 6, 10, 10, 9, 6, 3, 1)$.
- (ii) Let $J = (x^3, y^7, z^7, xy^2, xz^2)$. Then Condition 9.1 is satisfied if and only if $d = 5$ or $d = 6$. Note that R/J has a non-unimodal Hilbert function, $(1, 3, 6, 7, 6, 6, 7, 6, 5, 4, 3, 2, 1)$.
- (iii) Let $J' = (x^2, y^4, z^4, xy, xz)$. Then $d = 3$ satisfies Condition 9.1. Moreover, J' has a *non-strict* unimodal Hilbert function $(1, 3, 3, 4, 3, 2, 1)$.

Using Theorem 9.2, we easily recover [3, Theorem 6.2], one of the main results in the recent memoir [3].

Corollary 9.5. *Let R/I be a Artinian monomial algebra of type 2 over a field of characteristic zero. Then R/I has the weak Lefschetz property.*

Proof. By Theorem 9.2, we know that if I has four generators, then R/I has the weak Lefschetz property. If I has five generators, then it suffices to show that Condition 9.1 is vacuous in this case. Indeed, since R/I is level, we have that $a - \alpha = b - \beta + c - \gamma$ by Lemma 9.1. This implies

$$\frac{a + \alpha + \beta + \gamma}{2} = \frac{2\alpha + b + c}{2} \geq \alpha + \min\{b, c\}.$$

Hence, no integer d satisfies Condition 9.1. \square

Moreover, in most of the cases when the weak Lefschetz property holds in characteristic zero, we can give a linear lower bound on the characteristics for which the weak Lefschetz property must hold. Note that while the proof of Theorem 9.2 relies on properties of the bi-adjacency matrix $Z(T)$, the following argument also uses Proposition 8.22, which is based on the evaluation of a certain lattice path matrix.

Corollary 9.6. *Let R/I be a Artinian monomial algebra of type 2. Suppose that R/I has the weak Lefschetz property in characteristic zero and that there is no integer d such that*

$$(9.3) \quad \max \left\{ \alpha, b, c, \frac{\alpha + b + c}{2} \right\} < d < \min \left\{ a + \beta, a + \gamma, \alpha + \beta + c, \frac{a + \alpha + \beta + c}{2} \right\}.$$

Then R/I has the weak Lefschetz property, provided K has characteristic $p \geq \frac{1}{2}(a + b + c)$.

Proof. We use the notation introduced in the proof of Theorem 9.2. Fix any integer $d \geq 2$. Recall that, possibly after reordering rows and columns, the bi-adjacency matrix of $T = T_d(I)$ has the form (see Equation (9.2))

$$Z = \begin{pmatrix} Z(T^u) & 0 \\ Y & Z(T^l) \end{pmatrix}.$$

By assumption, d does not satisfy Condition (9.1) nor (9.3). This implies that T has one of the following properties: (i) T^u or T^l is balanced, (ii) T^u and T^l are both \triangle -heavy, or (iii) T^u and T^l are both ∇ -heavy.

The matrices $Z(T^u)$ and $Z(T^l)$ have maximal rank by Proposition 8.22 if the characteristic of K is at least $\lfloor \frac{a-\alpha+\beta+c}{2} \rfloor$ and $\lfloor \frac{\alpha+b+c}{2} \rfloor$, respectively. Combining non-vanishing maximal minors of $Z(T^u)$ and $Z(T^l)$, it follows that the matrix Z has maximal rank as well if $\text{char } K \geq \frac{1}{2}(a + b + c)$. \square

In order to fully extend Theorem 9.2 to sufficiently large positive characteristics, it remains to consider the case where T^u is \triangle -heavy and T^l is ∇ -heavy. This is more delicate.

Example 9.7. Let $T = T_{10}(x^8, y^8, z^8, x^3y^5, x^3z^6)$ as in Figure 9.2, and let T' be the maximal minor given in Figure 9.2(i). In each lozenge tiling of T' , there is exactly one lozenge that crosses the splitting line. There are four possible locations for this lozenge; one of these is illustrated in Figure 9.2(ii). The enumeration of lozenge tilings of T' is thus the sum of the lozenge tilings with the lozenge in each of the four places along the splitting line. Each of the summands is the product of the enumerations of the resulting upper and lower regions.

In particular, we have that

$$\begin{aligned}
 |\det N(T')| &= 20 \cdot 60 + 45 \cdot 64 + 60 \cdot 60 + 50 \cdot 48 \\
 &= 2^4 \cdot 3 \cdot 5^2 + 2^6 \cdot 3^2 \cdot 5 + 2^4 \cdot 3^2 \cdot 5^2 + 2^5 \cdot 3 \cdot 5^2 \\
 &= 2^5 \cdot 3^2 \cdot 5 \cdot 7 \\
 &= 10080.
 \end{aligned}$$

Notice that while the four summands only have prime factors of 2, 3, and 5, the final enumeration also has a prime factor of 7.

Still, we can give a bound in this case, though we expect that it is very conservative. It provides the following extension of Theorem 9.2.

Proposition 9.8. *Let R/I be a Artinian monomial algebra of type 2 such that R/I has the weak Lefschetz property in characteristic zero. Then R/I has the weak Lefschetz property in positive characteristic, provided $\text{char } K \geq 3^e$, where $e = \frac{1}{2} \binom{\frac{1}{2}(a+b+c)+2}{2}$.*

This follows from Lemma 9.1 and the following general result, which provides an effective bound for Corollary 8.7 in the case of three variables.

Proposition 9.9. *Let R/I be any Artinian monomial algebra such that R/I has the weak Lefschetz property in characteristic zero. If I contains the powers x^a, y^b, z^c , then R/I has the weak Lefschetz property in positive characteristic whenever $\text{char } K > 3^{\frac{1}{2} \binom{\frac{1}{2}(a+b+c)+2}{2}}$.*

Proof. Define $I' = (x^a, y^b, z^c)$, and let d' be the smallest integer such that $0 \neq h_{R/I'}(d' - 1) \geq h_{R/I'}(d')$. Thus, $d' - 1 \leq \frac{1}{2}(a + b + c)$ by Lemma 8.21.

Let d be the smallest integer such that $0 \neq h_{R/I}(d - 1) \geq h_{R/I}(d)$. Then $d \leq d'$, as $I' \subset I$ and adding or enlarging punctures only exacerbates the difference in the number of upward- and downward-pointing triangles. Since R/I has the weak Lefschetz property in characteristic zero, the Hilbert function of R/I is strictly increasing up to degree $d - 1$. Hence, Proposition 8.4 implies that the degrees of non-trivial socle elements of R/I are at least $d - 1$. The socle of R/I is independent of the characteristic of K . Therefore Proposition 8.4 shows that, in any characteristic, R/I has the weak Lefschetz property if and only if the bi-adjacency matrices of $T_d(I)$ and $T_{d+1}(I)$ have maximal rank. Each row and column of a bi-adjacency matrix has at most three entries that equal one. All other entries are zero. Moreover the maximal square submatrices of $Z(T_d(I))$ and $Z(T_{d+1}(I))$ have at most $h_{R/I}(d - 1)$ rows. Since $h_{R/I}(d - 1) < h_R(d - 1) = \binom{d+1}{2} \leq 3e$, Hadamard's inequality shows that the absolute values of the maximal minors of $Z(T_d(I))$ and $Z(T_{d+1}(I))$, considered as integers, are less than 3^{2e} . Hence, any prime number $p \geq 3^e$ does not divide any of these non-trivial maximal minors. \square

As indicated above, we believe that the bound in Proposition 9.8 is far from being optimal. Through a great deal of computer experimentation, we offer the following conjecture.

Conjecture 9.10. *Let I be an Artinian monomial ideal in $R = K[x, y, z]$ such that R/I is of type two. If R/I has the weak Lefschetz property in characteristic zero, then R/I also has the weak Lefschetz property in characteristics $p > \frac{1}{2}(a + b + c)$.*

10. ARTINIAN MONOMIAL ALMOST COMPLETE INTERSECTIONS

We discuss another generalisation of monomial complete intersections. The latter have three minimal generators. This section presents an in-depth discussion of the Artinian monomial ideals with exactly four minimal generators. They are called Artinian monomial almost complete intersections. These ideals have been discussed, for example, in [8] and [45, Section 6]. In particular, we will answer some of the questions posed in [45]. Some of our results are used in [5] for studying ideals with the Rees property.

Each Artinian ideal of $K[x, y, z]$ with exactly four monomial minimal generators is of the form

$$I_{a,b,c,\alpha,\beta,\gamma} = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma),$$

where $0 \leq \alpha < a$, $0 \leq \beta < b$, and $0 \leq \gamma < c$, such that at most one of α , β , and γ is zero. If one of α , β , and γ is zero, then $R/I_{a,b,c,\alpha,\beta,\gamma}$ has type two. In this case, the presence of the weak Lefschetz property has already been described in Section 9, see in particular, Theorem 9.2 and Proposition 9.9. Thus, throughout this section we assume that the integers α , β , and γ are all positive.

10.1. Presence of the weak Lefschetz property.

We begin by recalling a few results. The first one shows that $R/I_{a,b,c,\alpha,\beta,\gamma}$ has type three. More precisely:

Proposition 10.1. [45, Proposition 6.1] *Let $I = I_{a,b,c,\alpha,\beta,\gamma}$ be defined as above. Then R/I has three minimal socle generators. They have degrees $\alpha + b + c - 3$, $a + \beta + c - 3$, and $a + b + \gamma - 3$.*

In particular, R/I is level if and only if $a - \alpha = b - \beta = c - \gamma$.

Brenner classified when the syzygy bundle of $I_{a,b,c,\alpha,\beta,\gamma}$ is semistable.

Proposition 10.2. [7, Corollary 7.3] *Let $I = I_{a,b,c,\alpha,\beta,\gamma}$ be defined as above, and suppose K is a field of characteristic zero. Set $d = \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$. Then I has a semistable syzygy bundle if and only if the following three conditions are satisfied:*

- (i) $\max\{a, b, c, \alpha + \beta + \gamma\} \leq d$;
- (ii) $\min\{\alpha + \beta + c, \alpha + b + \gamma, a + \beta + \gamma\} \geq d$; and
- (iii) $\min\{a + b, a + c, b + c\} \geq d$.

Furthermore, Brenner and Kaid showed that, for almost complete intersections, non-semistability implies the weak Lefschetz property in characteristic zero.

Proposition 10.3. [8, Corollary 3.3] *Let K be a field of characteristic zero. Then $I_{a,b,c,\alpha,\beta,\gamma}$ has the weak Lefschetz property if its syzygy bundle is not semistable.*

The conclusion of this result is not necessarily true in positive characteristic.

Example 10.4. Let $I = I_{5,5,3,1,1,2}$, and thus $d = 6$. Then the syzygy bundle of I is not semistable as $\alpha + \beta + c = 5 < d = 6$. However, the triangular region $T_6(I)$ is balanced and $\det Z(T_6(I)) = 5$. Hence, I does not have the weak Lefschetz property if and only if the characteristic of K is 5.

The following example illustrates that the assumption on the number of minimal generators cannot be dropped in Proposition 10.2.

Example 10.5. Consider the ideal $J = (x^5, y^5, z^5, xy^2z, xyz^2)$ with five minimal generators. Then Corollary 4.3 gives that the syzygy bundle of J is not semistable. Notice that $T_6(J)$ is balanced. However, $\det Z(T_6(J)) = 0$, and so R/J never has the weak Lefschetz property, regardless of the characteristic of K .

The number d in Proposition 10.2 is not assumed to be an integer. In fact, if it is not, then the algebra has the weak Lefschetz property.

Proposition 10.6. [45, Theorem 6.2] *Let K be a field of characteristic zero. Then $I_{a,b,c,\alpha,\beta,\gamma}$ has the weak Lefschetz property if $a + b + c + \alpha + \beta + \gamma \not\equiv 0 \pmod{3}$.*

Again, the conclusion of this result may fail in positive characteristic. Indeed, for the ideal $I_{5,5,3,1,1,2}$ in Example 10.4 we get $d = \frac{17}{3}$, but it does not have the weak Lefschetz property in characteristic 5.

The following result addresses the weak Lefschetz property in the cases that are left out by Propositions 10.3 and 10.6. Its first part extends [45, Lemma 7.1] from level to arbitrary monomial almost complete intersections. Observe that balanced triangular regions correspond to an equality of the Hilbert function in two consecutive degrees, dubbed “twin-peaks” in [45].

Proposition 10.7. *Let $I = I_{a,b,c,\alpha,\beta,\gamma}$, and assume $d = \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$ is an integer. If the syzygy bundle of I is semistable and d is integer, then $T_d(I)$ is perfectly-punctured and balanced.*

Moreover, in this case R/I has the weak Lefschetz property if and only if $\det Z(T_d(I))$ is not zero in K .

Proof. Note that condition (i) in Proposition 10.2 says that $T_d(I)$ has punctures of nonnegative side lengths $d - a, d - b, d - c$, and $d - (\alpha + \beta + \gamma)$. Furthermore, conditions (ii) and (iii) therein are equivalent to the fact that the degree of the least common multiple of any two of the minimal generators of I is at least d . Hence, Lemma 8.26 gives that the punctures of $T_d(I)$ do not overlap. Using the assumption that d is an integer, it follows that $T_d(I)$ is perfectly-punctured, and thus balanced.

Since the punctures of $T_d(I)$ do not overlap, the punctures of $T_{d-1}(I)$ are not overlapping nor touching. Using Corollary 8.27, we conclude that the degrees of the socle generators of R/I are at least $d - 2$. Hence, Corollary 8.13 gives that R/I has the weak Lefschetz property if and only if $\det Z(T_d(I))$ is not zero in K . \square

In the situation of Proposition 10.7, the fact that R/I has the weak Lefschetz property implies that $T_d(I)$ is tileable by Theorem 5.5. This combinatorial property remains true even if R/I fails to have the weak Lefschetz property.

Proposition 10.8. *Let $I = I_{a,b,c,\alpha,\beta,\gamma}$. If R/I fails to have the weak Lefschetz property in characteristic zero, then $d = \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$ is an integer and $T_d(I)$ is tileable.*

Proof. By Propositions 10.3 and 10.6, we know that the syzygy bundle of I is semistable and $d = \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$ is an integer. Hence by Proposition 10.7, $T_d(I)$ is perfectly-punctured. Now we conclude by Theorem 4.5. \square

Specialising results in Section 6 and 7, we can decide the presence of the weak Lefschetz property in almost all cases.

Theorem 10.9. *Let $I = I_{a,b,c,\alpha,\beta,\gamma} = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma)$ be an Artinian ideal with four minimal generators such that α , β , and γ are all positive. Assume the base field K has characteristic zero, and consider the following conditions:*

- (i) $\max\{a, b, c, \alpha + \beta + \gamma\} \leq d$;
- (ii) $\min\{\alpha + \beta + c, \alpha + b + \gamma, a + \beta + \gamma\} \geq d$;
- (iii) $\min\{a + b, a + c, b + c\} \geq d$; and
- (iv) $d = \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$ is an integer.

Then the following statements hold:

- (a) *If one of the conditions (i) - (iv) is not satisfied, then R/I has the weak Lefschetz property.*
- (b) *Assume all the conditions (i) - (iv) are satisfied. Then:*
 - (1) *The multiplication map $\times(x + y + z) : [R/I]_{j-2} \rightarrow [R/I]_{j-1}$ has maximal rank whenever $j \neq d$.*
 - (2) *The algebra R/I has the weak Lefschetz property if one of the following conditions is satisfied:*
 - (I) *Condition (ii) is an equality.*
 - (II) *$a + b + c + \alpha + \beta + \gamma$ is divisible by 6.*
 - (III) *$c = \frac{1}{2}(a + b + \alpha + \beta + \gamma)$.*
 - (IV) *The region $T_d(I)$ has an axes-central puncture (see Subsection 6.4) and one of $d - a, d - b, d - c$, and $d - (\alpha + \beta + \gamma)$ is not odd.*
 - (V) *$a = b$, $\alpha = \beta$, and c or γ is even.*
 - (3) *The algebra R/I fails to have the weak Lefschetz property if one of the following conditions is satisfied:*
 - (IV') *The region $T_d(I)$ has an axes-central puncture (see Subsection 6.4) and all of $d - a, d - b, d - c$, and $d - (\alpha + \beta + \gamma)$ are odd; or*
 - (V') *$a = b$, $\alpha = \beta$, and both c and γ are odd.*

Proof. Assertion (a) follows from Propositions 10.2, 10.3, and 10.6.

Consider now the claims in part (b). Then Proposition 10.7 gives that R/I has the weak Lefschetz property if and only if $\det Z(T_d(I))$ is not zero.

The assumptions in (b) guarantee that the punctures of $T = T_d(I)$ do not overlap and the degrees of the socle generators of R/I are at least $d - 2$. Then condition (I) implies that the puncture to the generator $x^\alpha y^\beta z^\gamma$ touches another puncture, whereas condition (II) says that this puncture has an even side length. In either case, R/I has the weak Lefschetz property by Proposition 6.15.

The proof of (b)(1) uses the Grauert-Mülich splitting theorem. We complete this part below Proposition 10.21.

The remaining assertions all follow from a result in Section 6 or 7, when combined with Proposition 10.7:

(III). The condition $c = \frac{1}{2}(a + b + \alpha + \beta + \gamma)$ is equivalent to $d - c = 0$. Thus Proposition 6.8 gives the claim.

(IV) and (IV'). Use Corollary 6.17.

(V) and (V'). Use Proposition 7.12 and Theorem 7.4. □

Notice that Theorem 10.9(b)(1) says that, for almost monomial complete intersections, the multiplication map can fail to have maximal rank in at most one degree.

- Remark 10.10.** (i) Theorem 10.9 can be extended to fields of sufficiently positive characteristic by using Proposition 9.9. This lower bound on the characteristic can be improved whenever we know the determinant of $Z(T_d(I))$ from a result in Section 6 or 7. We leave the details to the reader.
- (ii) Question 8.2(2c) in [45] asked if there exist non-level almost complete intersections which never have the weak Lefschetz property. The almost complete intersection $I = I_{3,5,5,1,2,2} = (x^3, y^5, z^5, xy^2z^2)$ is not level and never has the weak Lefschetz property, regardless of field characteristic, as $\det Z(T_6(I)) = 0$ by Theorem 7.4.

10.2. Level almost complete intersections.

In Subsection 6.4, we considered one way of centralising the inner puncture of a triangular region associated to a monomial almost complete intersection. We called such punctures “axes-central.” In this section, we consider another method of centralising the inner puncture of such a triangular region. It turns out this method of centralisation is equivalent to the algebra being level.

Consider the ideal $I = I_{a,b,c,\alpha,\beta,\gamma}$ as above. Let d be an integer and assume that $T = T_d(I)$ has one floating puncture. We say the inner puncture of T is a *gravity-central puncture* if the vertices of the puncture are each the same distance from the puncture opposite to it (see Figure 10.1).

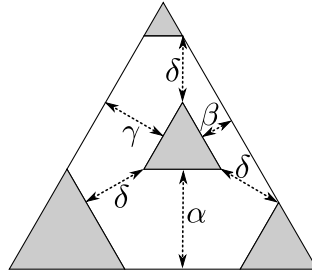


FIGURE 10.1. A prototypical figure with a gravity-central puncture.

Lemma 10.11. *Let $I = I_{a,b,c,\alpha,\beta,\gamma}$ such that $T_d(I)$ has a gravity-central puncture. Then R/I is a level type 3 algebra.*

Proof. The defining property for the distances is $(d-b) + (d-c) - \alpha = (d-a) + (d-c) - \beta = (d-a) + (d-b) - \gamma$. This is equivalent to the condition in Proposition 10.1 that R/I is level, i.e., $a - \alpha = b - \beta = c - \gamma$. \square

Since having a gravity-central puncture has an algebraic interpretation, it is natural to wonder if this is also true for the existence of an axes-central puncture.

Question 10.12. Let $I = I_{a,b,c,\alpha,\beta,\gamma}$. Does the existence of an axes-central puncture in $T_d(I)$ admit an algebraic characterisation?

Level almost complete intersections were studied extensively in [45, Sections 6 and 7]. In particular, Migliore, Miró-Roig, and the second author proposed a conjectured characterisation for the presence of the weak Lefschetz property for such algebras. We recall it here, though we present it in a different, but equivalent, form to better elucidate the reasoning behind it.

Conjecture 10.13. [45, Conjecture 6.8] *Let $I = I_{\alpha+t, \beta+t, \gamma+t, \alpha, \beta, \gamma}$ be an ideal of $R = K[x, y, z]$, where K has characteristic zero, $0 < \alpha \leq \beta \leq \gamma \leq 2(\alpha + \beta)$, $t \geq \frac{1}{3}(\alpha + \beta + \gamma)$, and $\alpha + \beta + \gamma$ is divisible by three. If $(\alpha, \beta, \gamma, t)$ is not $(2, 9, 13, 9)$ or $(3, 7, 14, 9)$, then R/I fails to have the weak Lefschetz property if and only if t is even, $\alpha + \beta + \gamma$ is odd, and $\alpha = \beta$ or $\beta = \gamma$. Furthermore, R/I fails to have the weak Lefschetz property in the two exceptional cases.*

The necessity part of this conjecture was proven in [45, Corollary 7.4]) by showing that R/I does not have the weak Lefschetz property if t is even, $\alpha + \beta + \gamma$ is odd, and $\alpha = \beta$ or $\beta = \gamma$. This result is covered by Theorem 10.9(b)(3)(V') because the region is mirror symmetric. It remained open to establish the presence of the weak Lefschetz property. Theorem 10.9 does this in many new cases.

Proposition 10.14. *Consider the ideal $I = I_{\alpha+t, \beta+t, \gamma+t, \alpha, \beta, \gamma}$ as given in Conjecture 10.13. Then R/I has the weak Lefschetz property if one of the following conditions is satisfied:*

- (i) t and $\alpha + \beta + \gamma$ have the same parity; or
- (ii) t is odd and $\alpha = \beta = \gamma$ is even.

Proof. We apply Theorem 10.9 with $d = t + \frac{2}{3}(\alpha + \beta + \gamma)$. Then the side length of the inner puncture of $T_d(I)$ is $t - \frac{1}{3}(\alpha + \beta + \gamma)$. Hence (i) follows from Theorem 10.9(b)(II). Claim (ii) is a consequence of Theorem 10.9(b)(IV) as the given condition implies the inner puncture is axes-central. \square

Remark 10.15. Conjecture 10.13 remains open in two cases, both of which are conjectured to have the weak Lefschetz property:

- (i) t even, $\alpha + \beta + \gamma$ is odd, and $\alpha < \beta < \gamma$; and
- (ii) t odd, $\alpha + \beta + \gamma$ is even, and $\alpha < \beta$ or $\beta < \gamma$.

Note that if true, then Conjecture 7.6 implies part (ii) in the case, where $\alpha = \beta$ or $\beta = \gamma$.

Notice that $T = T_d(I_{a,b,c,\alpha,\beta,\gamma})$ is simultaneously axis- and gravity-central precisely if either $a = b = c$ and $\alpha = \beta = \gamma$, or $a = b + 2 = c + 1$ and $\alpha = \beta + 2 = \gamma + 1$. In the former case, the weak Lefschetz property in characteristic zero is completely characterised below, strengthening [45, Corollary 7.6].

Corollary 10.16. *Let $I = I_{a,a,a,\alpha,\alpha,\alpha} = (x^a, y^a, z^a, x^\alpha, y^\alpha, z^\alpha)$, where $a > \alpha$. Then R/I fails to have the weak Lefschetz property in characteristic zero if and only if α and a are odd and $a \geq 2\alpha + 1$.*

Proof. If $a < 2\alpha$, then R/I has the weak Lefschetz property by Theorem 10.9(a).

Assume now $a \geq 2\alpha$. Then R/I fails the weak Lefschetz property if α and a are odd by [45, Corollary 7.6] (or Theorem 10.9(b)(3)(V')). Otherwise, R/I has this property by Proposition 10.14. \square

For $a \geq 2\alpha$, the triangular region $T_{a+\alpha}(I)$ was considered by Krattenthaler in [34]. He described a bijection between cyclically symmetric lozenge tilings of the region and descending plane partitions with specific conditions.

10.3. Splitting type and regularity.

The generic splitting type of a vector bundle on projective space is an important invariant. However, its computation is often challenging. In this section we consider the splitting type of the syzygy bundles of monomial almost complete intersections in R . These are rank three

bundles on the projective plane. For the remainder of this section we assume K is an infinite field.

Let $I = I_{a,b,c,\alpha,\beta,\gamma}$ as above. Recall from Section 4 that the syzygy module $\text{syz } I$ of I is defined by the exact sequence

$$0 \longrightarrow \text{syz } I \longrightarrow R(-\alpha - \beta - \gamma) \oplus R(-a) \oplus R(-b) \oplus R(-c) \longrightarrow I \longrightarrow 0,$$

and the syzygy bundle $\widetilde{\text{syz}} I$ on \mathbb{P}^2 of I is the sheafification of $\text{syz } I$. Its restriction to any line H of \mathbb{P}^2 splits as $\mathcal{O}_H(p) \oplus \mathcal{O}_H(q) \oplus \mathcal{O}_H(r)$. The triple (p, q, r) depends on the choice of the line H , but is the same for all general lines. This latter triple is called the *generic splitting type* of $\widetilde{\text{syz}} I$. Since I is a monomial ideal, the arguments in Proposition 8.6 imply that the generic splitting type (p, q, r) can be determined if we restrict to the line defined by $\ell = x + y + z$.

For computing the generic splitting type of $\widetilde{\text{syz}} I$, we use the observation that $R/(I, \ell) \cong S/J$, where $S = K[x, y]$, and $J = (x^a, y^b, (x + y)^c, x^\alpha y^\beta (x + y)^\gamma)$. Define an S -module $\text{syz } J$ by the exact sequence

$$(10.1) \quad 0 \longrightarrow \text{syz } J \longrightarrow S(-\alpha - \beta - \gamma) \oplus S(-a) \oplus S(-b) \oplus S(-c) \longrightarrow J \longrightarrow 0$$

using the, possibly non-minimal, set of generators $\{x^a, y^b, (x + y)^c, x^\alpha y^\beta (x + y)^\gamma\}$ of J . Then $\text{syz } J \cong S(p) \oplus S(q) \oplus S(r)$, where (p, q, r) is the generic splitting type of the vector bundle $\widetilde{\text{syz}} I$. The Castelnuovo-Mumford regularity of the ideal J is $\text{reg } J = 1 + \text{reg } S/J$.

For later use we record the following facts.

Remark 10.17. Adopt the above notation. Then the following statements hold:

- (i) Using, for example, the Sequence (10.1), one gets $-(p + q + r) = a + b + c + \alpha + \beta + \gamma$.
- (ii) If any of the generators of J is extraneous, then the degree of that generator is one of $-p$, $-q$, or $-r$.
- (iii) As the regularity of J is determined by the Betti numbers of S/J , we obtain that $\text{reg } J + 1 = \max\{-p, -q, -r\}$ if the Sequence (10.1) is a minimal free resolution of J .

Before moving on, we prove a technical but useful lemma.

Lemma 10.18. *Let $S = K[x, y]$, where K is a field of characteristic zero. Consider the ideal $\mathfrak{a} = (x^a, y^b, x^\alpha y^\beta (x + y)^\gamma)$ of S , and assume that the given generating set is minimal. Then $\text{reg } \mathfrak{a}$ is*

$$-1 + \max \left\{ a + \beta, b + \alpha, \min \left\{ a + b, a + \beta + \gamma, b + \alpha + \gamma, \left\lceil \frac{1}{2}(a + b + \alpha + \beta + \gamma) \right\rceil \right\} \right\}.$$

Proof. We proceed in three steps.

First, considering the minimal free resolution of the ideal $(x^a, y^b, x^\alpha y^\beta)$, we conclude

$$\text{reg}(x^a, y^b, x^\alpha y^\beta) = -1 + \max\{a + \beta, b + \alpha\}.$$

Second, the algebra $S/(x^a, y^b)$ has the strong Lefschetz property in characteristic zero (see, e.g., [28, Proposition 4.4]). Thus, the Hilbert function of $S/(x^a, y^b, (x + y)^\gamma)$ is

$$\dim_K [S/(x^a, y^b, (x + y)^\gamma)]_j = \max\{0, \dim_K [S/(x^a, y^b)]_j - \dim_K [S/(x^a, y^b)]_{j-\gamma}\}.$$

By analyzing when the difference becomes non-positive, we get that

$$(10.2) \quad \text{reg}(x^a, y^b, (x + y)^\gamma) = -1 + \min \left\{ a + b, a + \gamma, b + \gamma, \left\lceil \frac{1}{2}(a + b + \gamma) \right\rceil \right\}.$$

Third, notice that

$$(x^a, y^b, x^\alpha y^\beta (x+y)^\gamma) : x^\alpha y^\beta = (x^{a-\alpha}, y^{b-\beta}, (x+y)^\gamma).$$

Hence, multiplication by $x^\alpha y^\beta$ induces the short exact sequence

$$0 \rightarrow [S/(x^{a-\alpha}, y^{b-\beta}, (x+y)^\gamma)](-\alpha - \beta) \xrightarrow{\times x^\alpha y^\beta} S/\mathfrak{a} \rightarrow S/(x^a, y^b, x^\alpha y^\beta) \rightarrow 0.$$

It implies

$$\operatorname{reg} \mathfrak{a} = \max\{\alpha + \beta + \operatorname{reg}(x^{a-\alpha}, y^{b-\beta}, (x+y)^\gamma), \operatorname{reg}(x^a, y^b, x^\alpha y^\beta)\}.$$

Using the first two steps, the claim follows. \square

Recall that Proposition 10.2 gives a characterisation of the semistability of the syzygy bundle $\widetilde{\operatorname{syz}} I_{a,b,c,\alpha,\beta,\gamma}$, using only the parameters a, b, c, α, β , and γ . We determine the splitting type of $\widetilde{\operatorname{syz}} I_{a,b,c,\alpha,\beta,\gamma}$ for the nonsemistable and the semistable cases separately.

10.3.1. Nonsemistable syzygy bundle.

We first consider the case when the syzygy bundle is not semistable, and therein we distinguish four cases. It turns out that in three cases, at least one of the generators of the ideal J is extraneous.

Proposition 10.19. *Consider the ideal $I = I_{a,b,c,\alpha,\beta,\gamma} = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma)$ with four minimal generators. Assume that the base field K has characteristic zero and, without loss of generality, that $a \leq b \leq c$. Set $d := \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$, and denote by (p, q, r) the generic splitting type of $\widetilde{\operatorname{syz}} I$. Assume that $\widetilde{\operatorname{syz}} I$ is not semistable. Then:*

(i) *If $\min\{\alpha + \beta + \gamma, c\} \geq a + b - 1$, then*

$$(p, q, r) = (-c, -\alpha - \beta - \gamma, -a - b).$$

(ii) *Assume $\min\{\alpha + \beta + \gamma, c\} \leq a + b - 2$ and*

$$\frac{1}{2}(a + b + c) \leq \min \left\{ a + \beta + \gamma, b + \alpha + \gamma, c + \beta + \gamma, \frac{1}{2}(a + b + \alpha + \beta + \gamma) \right\}.$$

Then

$$(p, q, r) = (-\alpha - \beta - \gamma, -\left\lceil \frac{1}{2}(a + b + c) \right\rceil, -\left\lfloor \frac{1}{2}(a + b + c) \right\rfloor).$$

(iii) *Assume $\min\{\alpha + \beta + \gamma, c\} \leq a + b - 2$ and*

$$\frac{1}{2}(a + b + \alpha + \beta + \gamma) \leq \min \left\{ a + \beta + \gamma, b + \alpha + \gamma, c + \beta + \gamma, \frac{1}{2}(a + b + c) \right\}.$$

Then

$$(p, q, r) = (-c, q, -a - b - \alpha - \beta - \gamma + q),$$

where $-q = \min \{a + \beta + \gamma, b + \alpha + \gamma, \lceil \frac{1}{2}(a + b + \alpha + \beta + \gamma) \rceil\}$.

(iv) *Assume $\min\{\alpha + \beta + \gamma, c\} \leq a + b - 2$ and*

$$-r = \min \{a + \beta + \gamma, b + \alpha + \gamma, c + \beta + \gamma\} <$$

$$\min \left\{ \frac{1}{2}(a + b + \alpha + \beta + \gamma), \frac{1}{2}(a + b + c) \right\}.$$

Then

$$(p, q, r) = \left(\left\lfloor \frac{1}{2}(-3d - r) \right\rfloor, \left\lceil \frac{1}{2}(-3d - r) \right\rceil, r \right).$$

Proof. Set

$$\mu = \min \left\{ a + b, a + \beta + \gamma, b + \alpha + \gamma, c + \beta + \gamma, \frac{1}{2}(a + b + \alpha + \beta + \gamma), \frac{1}{2}(a + b + c) \right\}.$$

Using $a \leq b \leq c$, [7, Theorem 6.3] implies that the maximal slope of a subsheaf of $\widetilde{\text{syz}}I$ is $-\mu$. Since $\widetilde{\text{syz}}I$ is not semistable, we have $\mu < d$ (see Proposition 10.2). Moreover, the generic splitting type of $\widetilde{\text{syz}}I$ is determined by the minimal free resolution of $J = (x^a, y^b, (x + y)^c, x^\alpha y^\beta (x + y)^\gamma)$ as a module over $S = K[x, y]$. We combine both approaches to determine the generic splitting type.

Since $\text{reg}(x^a, y^b) = a + b - 1$, all polynomials in S whose degree is at least $a + b - 1$ are contained in (x^a, y^b) . Hence, $J = (x^a, y^b)$ if $\min\{\alpha + \beta + \gamma, c\} \geq a + b - 1$, and the claim in case (i) follows by Remark 10.17.

For the remainder of the proof, assume $\min\{\alpha + \beta + \gamma, c\} \leq a + b - 2$. Then $a + b > \frac{1}{2}(a + b + c)$, and thus $\mu \neq a + b$.

In case (ii), it follows that $\mu = \frac{1}{2}(a + b + c)$ and $c \leq \alpha + \beta + \gamma$, and thus $c \leq a + b - 2$. Using Equation (10.2), we conclude that

$$\text{reg}(x^a, y^b, (x + y)^c) = -1 + \min \left\{ a + b, \left\lceil \frac{1}{2}(a + b + c) \right\rceil \right\} = -1 + \left\lceil \frac{1}{2}(a + b + c) \right\rceil.$$

Observe now that $d > \mu = \frac{1}{2}(a + b + c)$ is equivalent to $\alpha + \beta + \gamma > \frac{1}{2}(a + b + c)$. This implies $\alpha + \beta + \gamma > \text{reg}(x^a, y^b, (x + y)^c)$, and thus $J = (x^a, y^b, (x + y)^c)$. Using Remark 10.17 again, we get the generic splitting type of $\widetilde{\text{syz}}I$ as claimed in (ii).

Consider now case (iii). Then $d > \mu = \frac{1}{2}(a + b + \alpha + \beta + \gamma)$, which gives $c > \frac{1}{2}(a + b + \alpha + \beta + \gamma)$. The second assumption in this case also implies $\frac{1}{2}(a + b + \alpha + \beta + \gamma) \leq a + \beta + \gamma$, which is equivalent to $b + \alpha \leq a + \beta + \gamma$ and also to $b + \alpha \leq \frac{1}{2}(a + b + \alpha + \beta + \gamma)$. Similarly, we have that $\frac{1}{2}(a + b + \alpha + \beta + \gamma) \leq b + \alpha + \gamma$, which is equivalent to $a + \beta \leq b + \alpha + \gamma$ and also to $a + \beta \leq \frac{1}{2}(a + b + \alpha + \beta + \gamma)$. It follows that

$$\max\{a + \beta, b + \alpha\} \leq \min \left\{ a + \beta + \gamma, b + \alpha + \gamma, \frac{1}{2}(a + b + \alpha + \beta + \gamma) \right\}.$$

Hence Lemma 10.18 yields

$$\begin{aligned} \text{reg}(x^a, y^b, x^\alpha y^\beta (x + y)^\gamma) = \\ -1 + \min \left\{ a + \beta + \gamma, b + \alpha + \gamma, \left\lceil \frac{1}{2}(a + b + \alpha + \beta + \gamma) \right\rceil \right\} < c. \end{aligned}$$

This shows that $(x + y)^c \in (x^a, y^b, x^\alpha y^\beta (x + y)^\gamma) = J$. Setting $-q = 1 + \text{reg } J$, Remark 10.17 provides the generic splitting type in case (iii).

Finally consider case (iv). Then $\mu = -r$, and μ is equal to the degree of the least common multiple of two of the minimal generators of I . In fact, $-\mu = r$ is the slope of the syzygy bundle $\mathcal{O}_{\mathbb{P}^2}(r)$ of the ideal generated by these two generators. Thus, the Harder-Narasimhan filtration (see [29, Definition 1.3.2]) gives an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(r) \rightarrow \widetilde{\text{syz}}I \rightarrow \mathcal{E} \rightarrow 0,$$

where \mathcal{E} is a semistable torsion-free sheaf on \mathbb{P}^2 of rank two and first Chern class $-a - b - c - \alpha - \beta - \gamma - r = -3d - r$. Its bidual \mathcal{E}^{**} is a stable vector bundle. Thus, by the theorem of Grauert and Müllich (see [25] or [51, Corollary 1 of Theorem 2.1.4]), its generic splitting type is $(\lfloor \frac{1}{2}(-3d - r) \rfloor, \lceil \frac{1}{2}(-3d - r) \rceil)$. Now the claim follows by restricting the above sequence to a general line of \mathbb{P}^2 . \square

We have seen that the ideal $J = (x^a, y^b, (x + y)^c, x^\alpha y^\beta (x + y)^\gamma)$ has at most three minimal generators in the cases (i) - (iii) of the above proposition. In the fourth case, the associated ideal $J \subset S$ may be minimally generated by four polynomials.

Example 10.20. Consider the ideal

$$I = I_{4,5,5,3,1,1} = (x^4, y^5, z^5, x^3 y z).$$

Then the corresponding ideal J is minimally generated by $x^4, y^5, (x + y)^5$, and $x^3 y (x + y)$. The syzygy bundle of $\widetilde{\text{syz}} I$ is not semistable, and its generic splitting type is $(-7, -6, -6)$ by Proposition 10.19(iv).

10.3.2. Semistable syzygy bundle.

Order the entries of the generic splitting type (p, q, r) of the semistable syzygy bundle $\widetilde{\text{syz}} I$ such that $p \leq q \leq r$. In this case, the splitting type determines the presence of the weak Lefschetz property (see [8, Theorem 2.2]). The following result is slightly more precise.

Proposition 10.21. *Let K be a field of characteristic zero, and assume the ideal $I = I_{a,b,c,\alpha,\beta,\gamma}$ has a semistable syzygy bundle. Set $k = \lfloor \frac{1}{3}(a + b + c + \alpha + \beta + \gamma) \rfloor$. Then the generic splitting type of $\widetilde{\text{syz}} I$ is*

$$(p, q, r) = \begin{cases} (-k - 1, -k, -k) & \text{if } a + b + c + \alpha + \beta + \gamma = 3k + 1; \\ (-k - 1, -k - 1, -k) & \text{if } a + b + c + \alpha + \beta + \gamma = 3k + 2; \\ (-k, -k, -k) & \text{if } a + b + c + \alpha + \beta + \gamma = 3k \text{ and} \\ & R/I \text{ has the weak Lefschetz property;} \\ (-k - 1, -k, -k + 1) & \text{if } a + b + c + \alpha + \beta + \gamma = 3k \text{ and} \\ & R/I \text{ fails to have the weak Lefschetz property.} \end{cases}$$

Proof. The Grauert-Müllich theorem [25] gives that $r - q$ and $q - p$ are both nonnegative and at most 1. Moreover, p, q , and r satisfy $a + b + c + \alpha + \beta + \gamma = -(p + q + r)$ (see Remark 10.17(i)). This gives the result if $k \neq d = \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$.

It remains to consider the case when $k = d$. Then $(-k, -k, -k)$ and $(-k - 1, -k, -k + 1)$ are the only possible generic splitting types. By Proposition 10.2(i), the minimal generators of the ideal $J = (x^a, y^b, (x + y)^c, x^\alpha y^\beta (x + y)^\gamma)$ have degrees that are less than d . Hence $\text{reg } J = d$ if and only if the splitting type of $\widetilde{\text{syz}} I$ is $(-d - 1, -d, -d + 1)$. Since $\dim_K [R/I]_{d-2} = \dim_K [R/I]_{d-1}$, using Proposition 10.7, we conclude that $\text{reg } J \geq d$ if and only if R/I does not have the weak Lefschetz property. \square

We are ready to add the missing piece in the proof of Theorem 10.9.

Completion of the proof of Theorem 10.9(b)(1).

We have just seen that the ideal $J = (x^a, y^b, (x + y)^c, x^\alpha y^\beta (x + y)^\gamma)$ has regularity d if R/I fails the weak Lefschetz property. This implies that the multiplication map $\times(x + y + x) :$

$[R/I]_{j-2} \rightarrow [R/I]_{j-1}$ is surjective whenever $j > d$. Moreover, since the minimal generators of J have degrees that are less than d , the exact sequence

$$0 \longrightarrow S(-d+1) \oplus S(-d) \oplus S(-d-1) \longrightarrow S(-\alpha-\beta-\gamma) \oplus S(-a) \oplus S(-b) \oplus S(-c) \longrightarrow J \longrightarrow 0$$

shows that $\dim_K[S/J]_{d-2} = 3$.

In the above proof of Theorem 10.9 we saw that the four punctures of $T_d(I)$ do not overlap and that $T_d(I)$ is balanced. Hence $T_{d-1}(I)$ has 3 more downward-pointing than upward-pointing triangles, that is,

$$\dim_K[R/I]_{d-2} = \dim_K[R/I]_{d-3} + 3.$$

It follows that the multiplication map in the exact sequence

$$[R/I]_{d-3} \longrightarrow [R/I]_{d-2} \longrightarrow S/J \longrightarrow 0$$

is injective. Since the degrees of the socle generators of R/I are at least $d-2$, Corollary 8.3 gives that $\times(x+y+z) : [R/I]_{j-2} \rightarrow [R/I]_{j-1}$ is injective whenever $j \leq d-1$. \square

The second author would like to thank the authors of [5]; it was during a conversation in the preparation of that paper that he learned about the use of the Grauert-Mulich theorem for deducing the injectivity of the map $[R/I]_{d-3} \rightarrow [R/I]_{d-2}$ in the above argument.

Example 10.22. Consider the ideal $I_{7,7,7,3,3,3} = (x^7, y^7, z^7, x^3y^3z^3)$ which never has the weak Lefschetz property, by Theorem 10.9(vii). The generic splitting type of $\widetilde{\text{syz}}I_{7,7,7,3,3,3}$ is $(-11, -10, -9)$. Notice that the similar ideal $I_{6,7,8,3,3,3} = (x^6, y^7, z^8, x^3y^3z^3)$ has the weak Lefschetz property in characteristic zero as $\det N_{6,7,8,3,3,3} = -1764$. The generic splitting type of $\widetilde{\text{syz}}I_{6,7,8,3,3,3}$ is $(-10, -10, -10)$.

We summarise part of our results for the case where I is associated to a tileable triangular region. Then the weak Lefschetz property is equivalent to several other conditions.

Theorem 10.23. *Let $I = I_{a,b,c,\alpha,\beta,\gamma} \subset R = K[x, y, z]$, where K is an infinite field of arbitrary characteristic. Assume I satisfies conditions (i)–(iv) in Theorem 10.9 and $d := \frac{1}{3}(a+b+c+\alpha+\beta+\gamma)$ is an integer. Then the following conditions are equivalent:*

- (i) *The algebra R/I has the weak Lefschetz property.*
- (ii) *The determinant of $N(T_d(I))$ (i.e., the enumeration of signed lozenge tilings of $T_d(I)$) is not zero in K .*
- (iii) *The determinant of $Z(T_d(I))$ (i.e., the enumeration of signed perfect matchings of $G(T_d(I))$) is not zero in K .*
- (iv) *The generic splitting type of $\widetilde{\text{syz}}I$ is $(-d, -d, -d)$.*

Proof. Regardless of the characteristic of K , the arguments for Proposition 10.7 show that $T_d(I)$ is balanced. Moreover, the degrees of the socle generators of R/I are at least $d-2$ as shown in Theorem 10.9(b)(1). Hence, Corollary 8.5 gives that R/I has the weak Lefschetz property if and only if the multiplication map

$$\times(x+y+z) : [R/I]_{d-2} \rightarrow [R/I]_{d-1}$$

is bijective. Now, Corollary 8.12 yields the equivalence of Conditions (i) and (ii). The latter is equivalent to condition (iii) by Theorem 5.17.

As above, let (p, q, r) be the generic splitting type of $\widetilde{\text{syz}}I$, where $p \leq q \leq r$, and let $J \subset S$ be the ideal such that $R/(I, x+y+z) \cong S/J$. The above multiplication map is bijective if and only if $\text{reg } J = d-1$. Since $\text{reg } J + 1 = -r$ and $p+q+r = -3d$, it follows

that $\text{reg } J = d - 1$ if and only if $(p, q, r) = (-d, -d, -d)$. Hence, conditions (i) and (iv) are equivalent. \square

11. FAILURE OF THE WEAK LEFSCHETZ PROPERTY

In this section we provide examples of Artinian monomial ideals that fail to have the weak Lefschetz property in various ways. In particular, in Subsection 11.2 we construct families of triangular regions (hence ideals) where the rank of the bi-adjacency matrix $Z(T)$ can be made as far from maximal as desired. In Subsection 11.3 we give examples of triangular regions such that the determinant of $Z(T)$ has large prime divisors, relative to the side length of T . That is, we offer examples of ideals that fail to have the weak Lefschetz property in large prime characteristics relative to the degrees of the generators.

As preparation, in Subsection 11.1 we prove that for checking maximal rank it is enough to consider triangular regions with only unit punctures.

11.1. Reduction to unit punctures.

We show that each triangular region T is contained in a triangular region \hat{T} such that \hat{T} only has unit punctures and $Z(T)$ has maximal rank if and only if $Z(\hat{T})$ has maximal rank. To see this we first replace a large puncture (side length at least two) by two non-overlapping subpunctures, one of which is a unit puncture. We need a partial extension of Proposition 6.1.

Proposition 11.1. *Let U be a balanced subregion of a triangular region T such that no downward-pointing unit triangle in U is adjacent to an upward-pointing unit triangle of $T \setminus U$. Then the following statements are true:*

- (i) *Possibly after reordering rows and columns of the bi-adjacency matrix of T , $Z(T)$ becomes a block matrix of the form*

$$\begin{pmatrix} Z(T \setminus U) & X \\ 0 & Z(U) \end{pmatrix}.$$

- (ii) *If $\det Z(U) \neq 0$ in K , then the bi-adjacency matrix $Z(T \setminus U)$ has maximal rank if and only if $Z(T)$ has maximal rank.*

Proof. The second assertion follows from the first one. The first assertion is immediate from the definition of the bi-adjacency matrix $Z(T)$. \square

Remark 11.2. If U is uniquely tileable, then Proposition 5.3 and Theorem 5.5 show that $\text{perm } Z(U) = |\det Z(U)| = 1$. Thus, $\det Z(U) \neq 0$, regardless of the characteristic of the base field K .

We are ready to describe the inductive step of our reduction to the case of unit punctures.

Lemma 11.3. *Let $T \subsetneq \mathcal{T}_d$ be a triangular region such that the sum of the side lengths of the non-unit punctures of T is $m > 0$. Then there exists a triangular region $\hat{T} \subset \mathcal{T}_d$ containing T such that the following statements are true.*

- (i) *The sum of the side lengths of the non-unit punctures of \hat{T} is at most $m - 1$.*
- (ii) *$\#\triangle(\hat{T}) - \#\nabla(\hat{T}) = \#\triangle(T) - \#\nabla(T)$.*
- (iii) *The bi-adjacency matrix $Z(T)$ has maximal rank if and only if $Z(\hat{T})$ has maximal rank.*

Proof. Among the bottom rows of a puncture of T whose side length is at least two, consider the row that is closest to the bottom of \mathcal{T}_d . In this row, pick a maximal *strip* S of *unit punctures*, that is, a sequence of adjacent upward- and downward-pointing triangles that all belong to punctures of T such that the downward-pointing triangles that are possibly adjacent to the left and right of S do not belong to a puncture of T . By the choice of the row, the strip S contains at least one downward-pointing triangle. Let P be any of the upward-pointing unit triangles in S . Denote by $U_1, U_2 \subset S$ the regions that are formed by the triangles in S to the left and to the right of P , respectively. Set $\hat{T} = T \cup U_1 \cup U_2$.

By construction, the downward-pointing triangles in $U_1 \cup U_2$ are not adjacent to any upward-pointing triangle in T . Furthermore, U_1 and U_2 are uniquely tileable. Thus, $1 = |\det Z(U_1)| = |\det Z(U_2)|$. Hence applying Proposition 11.1 twice, first to $U_1 \subset \hat{T}$ and then to $U_2 \subset \hat{T} \setminus U_1$, our assertions follow. \square

Notice that instead of using a row in the above argument, one can also use a maximal strip that is parallel and closest to the North-East or North-West boundary of \mathcal{T}_d . This follows from either the above arguments or a suitable rotation of the region T (compare Remark 5.10).

Repeating the procedure described in the preceding proof until the sum of the side lengths of the non-unit punctures is zero yields a triangular region \hat{T} containing T with the following properties:

- (i) \hat{T} has only unit punctures;
- (ii) $\#\triangle(\hat{T}) - \#\nabla(\hat{T}) = \#\triangle(T) - \#\nabla(T)$; and
- (iii) $Z(T)$ has maximal rank if and only if $Z(\hat{T})$ has maximal rank.

We call \hat{T} a *reduction of T to unit punctures*.

Example 11.4. The triangular region $T = T_8(x^7, y^7, z^6, xy^4z^2, x^3yz^2, x^4yz)$ and a reduction \hat{T} of T to unit punctures are depicted below. Notice that care must be taken when punctures overlap.

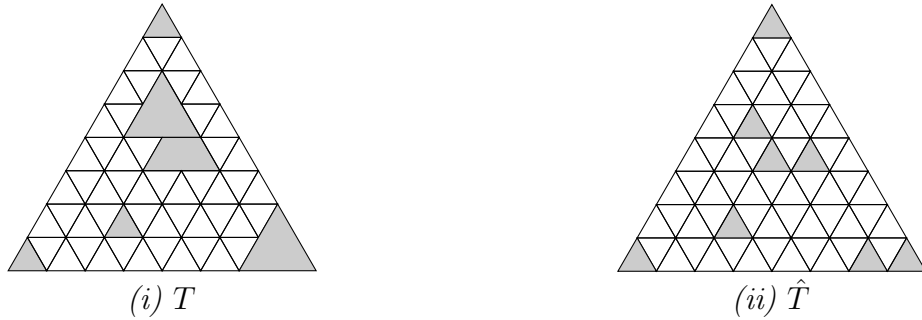


FIGURE 11.1. The region T as in Figure 2.5(i) and a reduction \hat{T} to unit punctures.

As pointed out above, Lemma 11.3 provides the following result.

Proposition 11.5. *Let $T \subsetneq \mathcal{T}_d$ be a triangular region. Then T has a reduction to unit punctures.*

Furthermore, $Z(T)$ has maximal rank if and only if $Z(\hat{T})$ has maximal rank for some (hence any) reduction \hat{T} of T to unit punctures.

In order to state this result algebraically and more precisely, we refine the definition of the weak Lefschetz property.

Definition 11.6. A graded K -algebra A is said to *fail the weak Lefschetz property in degree $d - 1$ by δ* if, for a general linear element $\ell \in [A]_1$, the rank of the multiplication map $\times \ell : [A]_{d-1} \rightarrow [A]_d$ is $r - \delta$, where $r = \min\{h_A(d - 1), h_A(d)\}$ is the expected rank.

Thus, “failing” the weak Lefschetz property in degree $d - 1$ by 0 means that A has the weak Lefschetz property in degree $d - 1$.

Suppose I is a monomial ideal of $R = K[x, y, z]$. Then R/I (or I) fails the weak Lefschetz property in degree $d - 1$ if and only if $Z(T_{d+1}(I))$ fails to have maximal rank. This follows by Proposition 8.6 and Corollary 8.12. Hence Proposition 11.5 implies:

Corollary 11.7. *Let $I \subset R$ be a monomial ideal such that $[I]_d \neq 0$, where $d \geq 1$. Then there is an Artinian ideal $J \subset I$ such that*

- (i) *The ideal J has no generators of degree less than d , that is, $[J]_{d-1} = 0$.*
- (ii) *The minimal generators of I and J whose degrees are at least $d + 1$ agree.*
- (iii) *If R/I is Artinian, then so is R/J .*
- (iv) *$h_{R/I}(d) - h_{R/I}(d - 1) = h_{R/J}(d) - h_{R/J}(d - 1)$.*
- (v) *R/I fails the weak Lefschetz property in degree $d - 1$ by δ if and only if R/J does.*

Proof. Let \hat{T} be a reduction of $T = T_{d+1}(I)$ to unit punctures. Then there is a unique monomial ideal $J' \subset I$ such that J' is generated in degree d and $\hat{T} = T_{d+1}(J')$. Let $J \subset I$ be the monomial ideal that is generated by the monomials in J' and the monomial minimal generators of I whose degree is greater than d . Then $\hat{T} = T_{d+1}(J)$. Since \hat{T} is a reduction of T to unit punctures, possibly after reordering rows and columns of its bi-adjacency matrix, $Z(\hat{T})$ becomes a block matrix of the form

$$\begin{pmatrix} Z(T) & X \\ 0 & Y \end{pmatrix},$$

where Y is a square matrix with $|\det Y| = 1$ (see Propositions 11.1 and 11.5). This implies all the assertions, except possibly (iv).

In order to address the latter, we use the flexibility of the procedure for producing a reduction to unit punctures. In fact, assume R/I is Artinian. Then I has minimal generators of the form x^a, y^b , and x^c . If $a < d$, $b < d$, or $c < d$, then we can choose a reduction of T to unit punctures such that the triangles labeled by x^d, y^d , and z^d , respectively, are punctures of $\hat{T} = T_{d+1}(J)$. Thus, $x^d \in J$, $y^d \in J$, or $z^d \in J$, respectively. \square

Observe that the assumption $[I]_d \neq 0$ in the above results is harmless. If $[I]_d = 0$, then R/I always has the weak Lefschetz property in degree $d - 1$.

11.2. Togliatti systems and Laplace equations.

Mezzetti, Miró-Roig, and Ottaviani showed in [43] that in some cases the failure of the weak Lefschetz property can be used to produce a variety satisfying a Laplace equation. Combined with our methods we produce toric surfaces that satisfy as many Laplace equations as desired.

We begin by reviewing the needed concepts from differential geometry. Throughout this section we assume that K is an algebraically closed field of characteristic zero. Let $X \subset$

$\mathbb{P}^N = \mathbb{P}_K^N$ be an n -dimensional projective variety, and let $P \in X$ be a smooth point. Choose affine coordinates and a local parametrisation φ around P , where $\varphi(0, \dots, 0) = P$ and the N components of φ are formal power series. Then the s -th osculating space $T_P^{(s)}X$ to X at P is the projectivised span of the partial derivatives of φ of order at most s . Its expected dimension is $\binom{n+s}{s} - 1$. The variety X is said to satisfy δ Laplace equations of order s if, for a general point P of X ,

$$\dim T_P^{(s)}X = \binom{n+s}{s} - 1 - \delta.$$

Remark 11.8. If $N < \binom{n+s}{s} - 1$, then X clearly satisfies at least one Laplace equation. However, this is not interesting.

There is a rich literature on varieties satisfying a Laplace equation (see, e.g., [61], [52], [43], [18] and the references therein). In [43], Mezzetti, Miró-Roig, and Ottaviani found a new approach to produce such varieties.

Let I be an ideal of $S = K[x_0, \dots, x_n]$ that is generated by forms f_1, \dots, f_r of degree d . Then I induces a rational map $\varphi_I : \mathbb{P}^n \dashrightarrow \mathbb{P}^{r-1}$ whose image we denote by $X_{n,[I]_d}$. It is a projection of the d -uple Veronese embedding of \mathbb{P}^n . Assume now that S/I is Artinian. Let $I^{-1} \subset S$ be the ideal generated by the Macaulay inverse system of I . The forms of degree d in I^{-1} induce a rational map $\varphi_{I^{-1}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\binom{n+d-1}{n}-r-1}$ whose image we denote by $X_{n,[I^{-1}]_d}$. Note that in the case, where $I \subset S$ is an Artinian monomial ideal, the ideal I^{-1} is generated by the monomials of S that are not in I . We now give a quantitative version of [43, Theorem 3.2], which directly follows from the arguments for the original statement.

Theorem 11.9. *Let I be an Artinian ideal of $S = K[x_0, \dots, x_n]$ that is minimally generated by $r \leq \binom{n+d-1}{n-1}$ forms of degree d . Then the following conditions are equivalent:*

- (i) *The ideal I fails the weak Lefschetz property in degree $d-1$ by δ .*
- (ii) *The variety $X_{n,[I^{-1}]_d}$ satisfies δ Laplace equations of order $d-1$.*

Note that the assumption $r \leq \binom{n+d-1}{n-1}$ simply ensures that the variety $X_{n,[I^{-1}]_d}$ does not trivially satisfy a Laplace equation in the sense of Remark 11.8. The assumption also implies that R/I has the weak Lefschetz property in degree $d-1$ if and only if the multiplication map $\times \ell : [R/I]_{d-1} \rightarrow [R/I]_d$ is injective.

Following [43], an ideal I is said to define a *Togliatti system* if it satisfies the two equivalent conditions in Theorem 11.9. The name is in honor of Togliatti who proved in [61] that the only example for $n=2$ and $d=3$ is $I = (x_0^3, x_1^3, x_2^3, x_0x_1x_2)$.

Now we restrict ourselves to the case of three variables, i.e., $n=2$. An Artinian monomial ideal I of $R = K[x, y, z]$ defines a Togliatti system if it is generated by $r \leq d+1$ monomials of degree d and it fails the weak Lefschetz property in degree $d-1$. This corresponds precisely to the triangular regions $T \subset \mathcal{T}_{d+1}$ with $r \leq d+1$ unit punctures, which include the three upward-pointing unit triangles in each corner of \mathcal{T}_{d+1} , and a bi-adjacency matrix $Z(T)$ that does not have maximal rank.

Hence, often a monomial ideal $I \subset R$ that fails the weak Lefschetz property in degree $d-1$ gives rise to a Togliatti system by using Proposition 11.5.

Theorem 11.10. *Let $I \subset R$ be an Artinian monomial ideal, generated by monomials of degree at most d . Assume $h_{R/I}(d-1) \leq h_{R/I}(d)$ and R/I fails the weak Lefschetz property in degree $d-1$ by $\delta > 0$. Then, for every reduction $\hat{T} = T_{d+1}(\hat{I})$ of $T = T_{d+1}(I)$ to unit*

punctures such that the ideal \hat{I} is Artinian, \hat{I} defines a Togliatti system. Moreover, the variety $X_{n, [\hat{I}^{-1}]_d}$ satisfies δ Laplace equations of order $d - 1$.

Proof. Combine Theorem 11.9 and Corollary 11.7. Note that the latter also guarantees that there is always at least one reduction such that \hat{I} is Artinian. \square

Observe that the above $X_{n, [\hat{I}^{-1}]_d}$ is a toric surface since \hat{I}^{-1} is a monomial ideal. All toric surfaces are quasi-smooth by [24, §5.2].

In order to illustrate the last result we exhibit a specific example.

Example 11.11. Let $I = (x^d, y^d, z^d, xyz)$, for some $d \geq 3$. Then $T = T_{d+1}(I)$ is a balanced region. Moreover, $|\det Z(T)| = 0$ if d is odd (see Proposition 10.14(i)) and $|\det Z(T)| = 2$ if d is even. This is also proven in [15, Proposition 3.1].

Suppose d is odd. Then the ideal $\hat{I} = (x^d, y^d, z^d, m_1, \dots, m_{d-1})$, where $m_i = x^i y^j z^{d-i-j}$ for some $1 \leq j \leq d - i - 1$, defines a Togliatti system as $T_{d+1}(\hat{I})$ is a reduction of T to unit punctures. It is obtained by picking an upward pointing triangle in each row of the puncture associated to xyz . Using instead a diagonal and two rows produces the region depicted in Figure 11.2. However, Figure 11.2(ii) is not formed by using the procedure described above

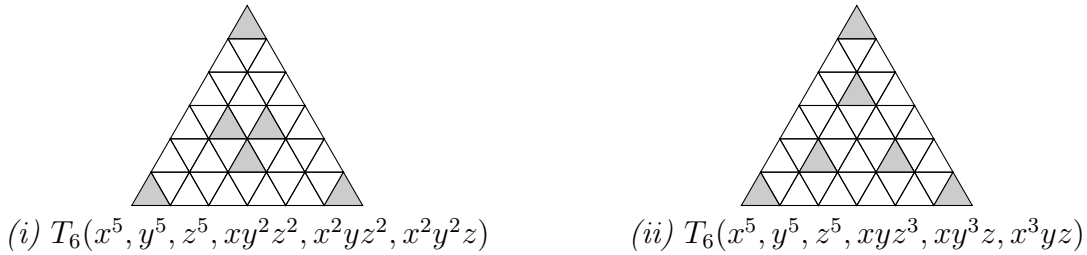


FIGURE 11.2. Triangular regions whose associated ideals define Togliatti systems. Both regions are formed by reducing $T_6(x^5, y^5, z^5, xyz)$.

Proposition 11.5. Instead, it is obtained by removing a tileable region from the central puncture.

We now describe the Artinian ideals of R with few generators that define Togliatti systems. No such ideal exists with three generators. Togliatti [61] proved that the only such ideal with four generators of degree three is (x^3, y^3, z^3, xyz) . Moreover, Mezzetti, Miró-Roig, and Ottaviani showed in [43, Theorem 5.1] that no ideal of R with four generators whose degree is at least four defines a Togliatti system. We now classify the Artinian monomial ideal ideals with five generators that define Togliatti systems.

Proposition 11.12. *Let $I \subset R$ be an Artinian ideal minimally generated by five monomials all of degree d . Then I defines a Togliatti system if and only if, up to a change of variables, I is either of the form $(x^4, y^4, z^4, x^2yz, y^2z^2)$ or $(x^d, y^d, z^d, x^{d-1}y, x^{d-1}z)$ with $d \geq 4$.*

Proof. Since I is Artinian, it must be of the form $I = (x^d, y^d, z^d, m, n)$, where m and n are monomials of degree d . The assumption forces $d \geq 2$.

If $m = x^{d-1}y$ and $n = x^{d-1}z$, then the residue class of x^{d-1} is a socle element of R/I . Hence, R/I fails the weak Lefschetz property in degree $d - 1$ by Proposition 8.4(ii), as claimed. This covers in particular the case $d = 2$.

Suppose $d = 7$. Then there are $\binom{7+2}{2} - 3 = 33$ monomials of degree d in $R \setminus \{x^d, y^d, z^d\}$. Thus, there are $\binom{33}{2} = 528$ choices for picking the two monomials $m \neq n$. Checking all these ideals using a computer yields that the claim is true if $d = 7$. Similarly, one checks that no such ideal exists if $3 \leq d \leq 6$.

Suppose $d \geq 8$. Write $m = x^\alpha y^\beta z^\gamma$ and $n = x^\alpha y^\beta z^\gamma$. We consider two cases.

Case 1: Assume $2 \leq a \leq \alpha$. By the beginning of the proof we may also assume $(a, \alpha) \neq (d-1, d-1)$. We choose in each row of $T = T_{d+1}$ above the row with the puncture associated to m an upward pointing triangle as a new puncture, with two exceptions. If $a \neq \alpha$, then we do not pick a puncture in the row that contains the puncture to the monomial n . If $a = \alpha$, then we do not pick a puncture in the row adjacent to the row containing the punctures to m and n . In both cases, we get a triangular region $T' \subset T$ that has $d - 2 - a$ more unit punctures than T . In fact, T' is a reduction to unit punctures of $T'' = T_{d+1}(x^a, y^d, z^d)$. By Proposition 8.22, $Z(T'')$ has maximal rank. (Recall that we assume $\text{char } K = 0$.) Hence, $Z(T')$ has maximal rank, and so does $Z(T)$ as $\det Z(T')$ is a maximal minor of $Z(T)$. This concludes the first case.

Case 1 takes care of the situations where both numbers in the pairs (a, α) , (b, β) , or (c, γ) are at least two. Thus, without loss of generality it remains to consider the following case.

Case 2: Assume $a, b \leq 1$ and $\gamma \leq 1$. Then we can pick as a new puncture a unit triangle whose label is a multiple of z^{d-2} such that the resulting region T' is a reduction of unit punctures of $T'' = T_{d+1}(x^d, y^d, z^{d-2}, n)$.

We may also assume that $\alpha \geq \beta$. Thus $\alpha \geq \frac{d-1}{2}$, so $\alpha \geq 4$. Now we pick a new puncture in each row of T'' above the row containing the puncture to n . Call the result \tilde{T} . Then \tilde{T} is a reduction to unit punctures of $T_{d+1}(x^\alpha, y^d, z^{d-2})$. Using again Proposition 8.22, it follows that $Z(\tilde{T})$, $Z(T'')$, $Z(T')$, and thus $Z(T)$ have maximal rank. \square

Remark 11.13. (i) The above result for $d = 4$ was shown independently by Di Genaro, Ilardi, and Vallès [18, Theorem 3.2].

(ii) Notice that $T = T_5(x^4, y^4, z^4, x^2yz, y^2z^2)$ is mirror symmetric with three odd axial punctures. Hence $\det Z(T) = 0$ by Theorem 7.4, giving a direct argument that the ideal $(x^4, y^4, z^4, x^2yz, y^2z^2)$ defines a Togliatti system.

In Example 11.11, we only showed that the bi-adjacency matrices do not have maximal rank. We now consider how much maximal rank can fail. To this end we construct ideals that have the weak Lefschetz property in all degrees, except $d - 1$, and give rise to varieties satisfying many Laplace equations of order $d - 1$.

We begin with a general construction that proves useful for generating such examples. We note that this construction is based on modifications of triangular regions that are similar to the techniques used in Subsection 11.1.

Proposition 11.14. *Let J be an Artinian ideal that is minimally generated by m monomials of degree $e \leq d$. Let j and k be integers such that*

$$1 \leq j \leq \min \left\{ \frac{d-1}{m}, \frac{d+1}{e+1} \right\} \quad \text{and} \quad 0 \leq k \leq d - mj - 1.$$

Define an ideal $I = I_{J,d,j,k}$ by

$$I_{J,d,j,k} = J \cdot x^{d+1-(e+1)j} \cdot (x^{e+1}, y^{e+1})^{j-1} + (y^d) + z^{mj+k+1} \cdot (y, z)^{d-1-mj-k}.$$

Then the following statements are true.

- (i) The ideal I is generated by $d + 1 - k$ monomials of degree d , so $T_{d+1}(I)$ has only unit punctures.
- (ii) For $i \in \{1, 2\}$, the rows and columns of $Z(T_{d+i}(I))$ can be rearranged so that it becomes a block matrix of the form

$$\begin{pmatrix} X_{0,i} & X_{1,i} & X_{2,i} & X_{3,i} & \cdots & X_{j,i} \\ 0 & Z(T_{e+i}(J)) & 0 & 0 & \cdots & 0 \\ 0 & 0 & Z(T_{e+i}(J)) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & Z(T_{e+i}(J)) & 0 \\ 0 & 0 & 0 & \cdots & 0 & Z(T_{e+i}(J)) \end{pmatrix},$$

where the matrix $X_{0,i}$ has maximal rank.

Proof. The definition of I gives that I is generated by $mj + 1 + d - mj - k = d + 1 - k$ monomials of degree d .

By Proposition 8.22, the bi-adjacency matrix of the complete intersection region $T'' = T_{d+1}(x^{d+1-(e+1)j}, y^d, z^{mj+k+1})$ has maximal rank. We compare this region with the region to the ideal

$$I' = x^{d+1-(e+1)j} \cdot (x^{e+1}, y^{e+1})^{j-1} + (y^d) + z^{mj+k+1} \cdot (y, z)^{d-1-mj-k}.$$

The region $T' = T_{d+1}(I')$ has j non-overlapping punctures of side length $e + 1$ along the top-left edge and $d - mj - k$ non-overlapping unit punctures along the bottom edge. It contains T'' . In fact, T'' can be obtained from T' by removing uniquely tileable regions, namely rhombi. Applying Proposition 11.1 repeatedly, we get that $Z_{d+1}(T')$ has maximal rank.

Replacing each puncture of side length $e + 1$ of T' by $T_{e+1}(J)$ produces the region $T = T_{d+1}(I)$. Each such replacement amounts to removing from $T_{e+1}(J)$ all the present triangles. (See Figure 11.3 for an illustration.) Thus, Proposition 11.1 applies again. This proves claim (ii) if $i = 1$. Observe that $X_{0,1} = T_{d+1}(I')$.

For $i = 2$, we argue similarly, using the fact that the overlapping subregions $T_{e+2}(J)$ of $T_{d+2}(I)$ overlap in a unit puncture. Here, the assumption that J is an Artinian ideal is important. It implies that the unit triangles in the corners of $T_{e+i}(J)$ are punctures. \square

Choosing a suitable ideal J in the above result we can construct the desired Togliatti systems.

Corollary 11.15. *Consider the ideal $J = (x^3, y^3, z^3, xyz)$. Let $d \geq 5$, j , and k be integers satisfying $1 \leq j \leq \frac{d-1}{4}$, and $0 \leq k \leq d - 4j - 1$. Then $Z(T_{d+1}(I_{J,d,j,k}))$ is $\binom{d+1}{2} \times ((\binom{d+1}{2} + k)$ matrix of rank $Z \leq \binom{d+1}{2} - j$, i.e., it fails to have maximal rank by j .*

Moreover, for $k = 0$, the bi-adjacency matrix $Z(T_{d+2}(I_{J,d,j,0}))$ has maximal rank.

Proof. Set $I = I_{J,d,j,k}$, and $T = T_{d+1}(I)$.

By Proposition 11.14(i), I is generated by $d + 1 - k$ monomials of degree d . Hence $h_{R/I}(d - 1) = \binom{d+1}{2}$ and $h_{R/I}(d) = \binom{d+2}{2} - (d + 1 - k) = \binom{d+1}{2} + k$, and so $Z(T)$ is a $\binom{d+1}{2} \times ((\binom{d+1}{2} + k)$ matrix. Since $Z(T_4(J))$ is a 6×6 matrix of rank $5 < 6$ and there are j copies of $Z(T_4(J))$ along the block diagonal of $Z(T)$ by Proposition 11.14(ii), it follows that $\text{rank } Z(T) = \binom{d+1}{2} - j$.

Suppose now $k = 0$, and consider $T' = T_{d+2}(I)$. Using the notation of Proposition 11.14, note that $X_{0,2} = Z(T_{d+2}(I'))$, where

$$I' = x^{d+1-4j} \cdot (x^4, y^4)^{j-1} + (y^d) + z^{4j+1} \cdot (y, z)^{d-1-4j}.$$

The bi-adjacency matrix of the complete intersection region $T'' = T_{d+2}(x^{d+1-4j}, y^d, z^{4j+1})$ has two more rows than columns. Since T'' is obtained from $T_{d+2}(I')$ by removing balanced regions, also $X_{0,2} = Z(T_{d+2}(I'))$ has two more rows than columns. The matrix $T_5(J)$ is a 6×3 matrix of maximal rank. Hence Proposition 11.14(ii) proves that $Z(T_{d+2}(I))$ has maximal rank as well. \square

We illustrate the regions involved in the last statement in a specific case.

Example 11.16. Let $I = I_{J,d,j,k}$ be an ideal as in the preceding corollary, where $d = 13$, $j = 2$, and $k = 0$. The related triangular regions are depicted below.

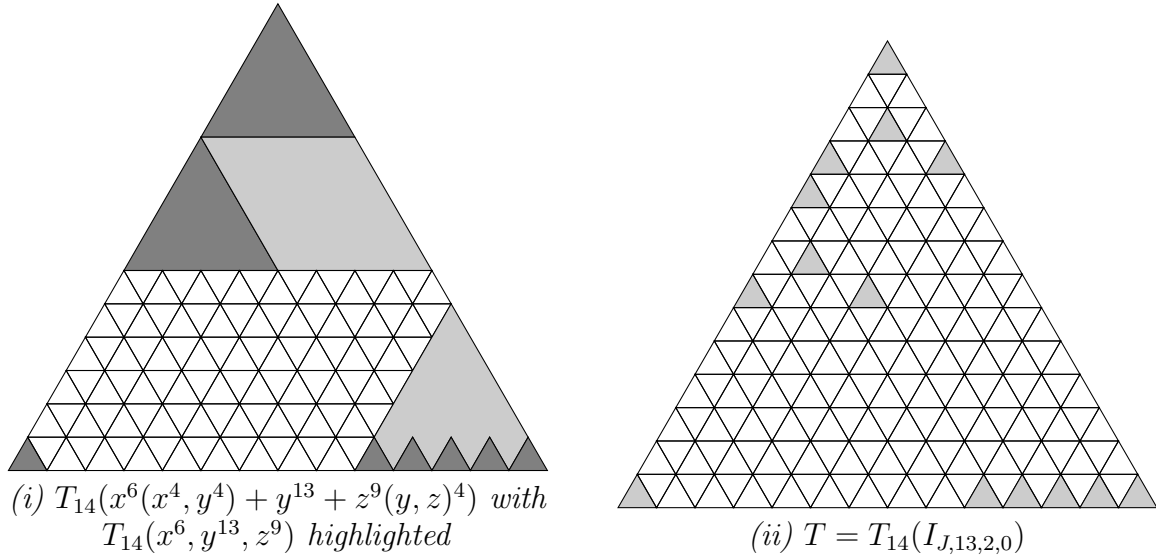


FIGURE 11.3. Construction of the region $T = T_{14}(I_{J,13,2,0})$, where $J = (x^3, y^3, z^3, xyz)$. The matrix $Z(T)$ fails to have maximal rank by 2.

Collecting the results about the ideals $I = I_{J,d,j,k}$ in the case $k = 0$, we get the following consequence.

Corollary 11.17. Let d and j be integers such that $1 \leq j \leq \frac{d-1}{4}$, and consider the ideal

$$I_j = (x^3, y^3, z^3, xyz) \cdot x^{d+1-4j} \cdot (x^4, y^4)^{j-1} + (y^d) + z^{4j+1}(y, z)^{d-1-4j}.$$

Then the following statements hold.

- (i) The algebra R/I_j has the weak Lefschetz property in every degree $i \neq d - 1$.
More precisely, for a general linear form ℓ , the map $\times \ell : [R/I_j]_{i-1} \rightarrow [R/I_j]_i$ is injective if $i < d$, and it is surjective if $j > d$.
- (ii) In degree $d - 1$, R/I_j fails the weak Lefschetz property by j .
- (iii) The variety $X_{n,[(I_j)^{-1}]_d}$ satisfies exactly j Laplace equations of order $d - 1$.

Proof. Combining Corollary 11.15 (with $k = 0$) and Proposition 8.6 shows assertions (i) and (ii). Notice that $\dim_K [R/I_j]_d = \dim_K [R/I_j]_{d+1} + 2$.

Now Theorem 11.10 gives (iii) because $\dim_K [R/I_j]_{d-1} = \dim_K [R/I_j]_d$. \square

It should be noted that one can produce more families with unexpected properties by varying the ideal J used in Proposition 11.14. We illustrate this by constructing a family of algebras that fail the weak Lefschetz property in two consecutive degrees, where in one degree injectivity was expected and in the other degree surjectivity was expected.

Corollary 11.18. *Let d, j and k be integers such that $1 \leq j \leq \frac{d-1}{6}$ and $0 \leq k \leq \min\{2j + 2, d - 6j - 1\}$. Consider the ideal*

$$I_{j,k} = J \cdot x^{d+1-4j} \cdot (x^4, y^4)^{j-1} + (y^d) + z^{6j+k+1} \cdot (y, z)^{d-1-6j-k},$$

where

$$J = (x^3, y^3, z^3, x^2y, xz^2, y^2z).$$

Then the algebra $R/I_{j,k}$ has the following properties.

- (i) $\dim_K[R/I_{j,k}]_d = \binom{d+1}{2} + k \geq \binom{d+1}{2} = \dim_K[R/I_{j,k}]_{d-1}$, and $R/I_{j,k}$ fails the weak Lefschetz property by at least $2j$.
- (ii) $\dim_K[R/I_{j,k}]_{d+1} = \dim_K[R/I_{j,k}]_d - (2j - k + 2) \leq \dim_K[R/I_{j,k}]_d$, and $R/I_{j,k}$ fails the weak Lefschetz property by $2j + k - 2$.

Proof. The arguments are similar to the ones used in the proof of Corollary 11.15. Thus, we mainly restrict ourselves to mentioning some of the differences.

We apply Proposition 11.14 with $e = 3$ and $m = 6$. Note that, for $i \in \{1, 2\}$, the matrix $X_{0,i}$ is $X_{0,i} = T_{d+i}(I')$, where

$$I' = x^{d+1-4j} \cdot (x^4, y^4)^{j-1} + (y^d) + z^{6j+1+k} \cdot (y, z)^{d-1-6j-k}.$$

The matrix $X_{0,1}$ has maximal rank and $2j + k$ more columns than rows. The 6×4 matrix $Z(T_4(J))$ has rank four. Hence, Proposition 11.14(ii) shows that $Z(T_{d+1}(I_{j,k}))$ fails to have maximal rank by at least $2j$. This proves claim (i).

Consider now the matrix $X_{0,2}$. It has maximal rank and $2j + k - 2$ more columns than rows. Since $Z(T_5(J))$ is a 4×0 matrix, it follows that the matrix $Z(T_{d+2}(I_{j,k}))$ is obtained from the matrix $X_{0,2}$ by appending $4j$ zero rows. Therefore, $Z(T_{d+1}(I_{j,k}))$ fails to have maximal rank by $2j + k - 2$. \square

Remark 11.19. The algebras in Corollaries 11.15 and 11.18 do not have low degree socle elements in the sense of Proposition 8.4(i). The degrees of their socle elements are at least d . Thus, the weak Lefschetz property fails for reasons *other* than having socle elements of small degree.

11.3. Large prime divisors.

Let $T = T_d(I)$ be a balanced triangular region, and consider $\det Z(T)$ as an integer. If it is not zero, then, by Corollary 8.12, the algebra R/I fails the weak Lefschetz property in degree $d - 1$ if and only if the characteristic $p > 0$ of the base field divides $\det Z(T)$. Throughout Sections 6–10, every time we are able to provide an upper bound on the prime divisors of $\det Z(T)$, it has been at most $d - 1$. However, this is not always true. We have numerous examples where, for small d , the prime factors of $\det Z(T)$ can be quite large relative to d .

Example 11.20. We provide Artinian monomial ideals I and integers d such that $0 \neq \det Z(T_d(I))$ has a prime divisor that is not less than d .

- (i) The smallest d such that $\det Z(T_d(I))$ is nonzero and divisible by some prime $p \geq d$ is $d = 5$. Indeed, if $I = (x^3, y^4, z^5, xz^3, y^2z^2)$, then $|\det Z(T_5(I))| = 5$.

- (ii) The smallest d such that $\det Z(T_d(I))$ is nonzero and divisible by some prime $p > d$ is $d = 6$. If $I = (x^5, y^5, z^5, x^3y^2, x^2z^3, y^3z^2)$, then $|\det Z(T_6(I))| = 35 = 5 \cdot 7$.
- (iii) A possibly surprising example is $I = (x^{20}, y^{20}, z^{20}, x^3y^8z^{13})$ with $d = 28$. In this case,

$$|\det Z(T_{28}(I))| = 2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 17^2 \cdot 19^6 \cdot 23^5 \cdot 20554657.$$

Here $\det Z(T_d(I))$ is divisible by a prime, 20554657, which is over 700,000 times larger than d .

- (iv) In the previous examples, the determinants have only a single prime factor that is larger than d . In general, there can be more such prime divisors. Indeed, let $I = (x^7, y^{12}, z^{15}, xy^7z^2)$ and $d = 14$. Then $|\det Z(T_{14}(I))| = 13 \cdot 17 \cdot 23$.

Note that, for a fixed integer d , there are finitely many Artinian monomial ideals whose generators have degrees at most d . We used *Macaulay2* [40] to search this finite space to find results such as (i) and (ii) above.

It would be desirable if, at least, there is a uniform upper bound on the prime divisors of $\det Z(T)$ that is linear in d . However, this does not appear to be the case in general. The following example suggests that some prime divisors of $\det Z(T)$ can be of the order d^2 .

Example 11.21. For $t \geq 4$, consider the level and type 3 algebra R/I_t , where

$$I_t = (x^{1+t}, y^{4+t}, z^{7+t}, xy^4z^7).$$

In [13, Section 6], Ciucu, Eisenkölbl, Krattenthaler, and Zare argue that the determinant of a punctured hexagonal subregion of \mathcal{T}_d is polynomial in d , if only the side length of the central puncture increases with d , whereas the size of the other punctures remains constant. Thus, one can use interpolation to determine this polynomial. Applying this procedure to $T_{t+8}(I_t)$, we get that $\det Z(T_{t+8}(I_t))$ is

$$\frac{4}{\mathcal{H}(7)} \cdot \begin{cases} (t-3)(t-2)(t-1)^3t^3(t+1)^2(t+2)(t+4)(t+6)(t^2+6t-1) & \text{if } t \text{ is odd;} \\ (t-2)^2(t-1)^2t^4(t+1)^2(t+2)(t+5)(t+7)(t^2+2t-9) & \text{if } t \text{ is even.} \end{cases}$$

We now recall a number-theoretic conjecture. Let $f \in \mathbb{Z}[t]$ be an irreducible polynomial whose degree is at least 2, and set $D = \gcd\{f(i) \mid i \in \mathbb{Z}\}$. In this case, Bouniakowsky conjectured in 1857 [6] that there are infinitely many integers t such that $\frac{1}{D}f(t)$ is a prime number. We note that the weaker Fifth Hardy-Littlewood conjecture is a special case of the Bouniakowsky conjecture. It posits that $t^2 + 1$ is prime for infinitely many positive integers t .

Observe that the quadratic factors of the above determinant, $t^2 + 6t - 1$ and $t^2 + 2t - 9$, respectively, are irreducible polynomials in $\mathbb{Z}[t]$. Thus, for infinitely many positive integers t , the above determinant has a prime divisor of order t^2 if Bouniakowsky's conjecture is true.

It follows that, assuming the Bouniakowsky conjecture, the above ideals provide regions $T_d \subset \mathcal{T}_d$ such that $\det Z(T_d) \neq 0$ has a prime divisor of order d^2 for infinitely many integers d .

Given the above examples, it seems unlikely that the prime divisors of $\det Z(T_d(I))$ are bounded linearly in d .

12. FURTHER OPEN PROBLEMS

In this closing section we wish to point out some additional questions and problems that are suggested by this work.

In Section 5 we introduced for *every* non-empty subregion $T \subset \mathcal{T}_d$ its bi-adjacency matrix $Z(T)$ and its perfect matching matrix $N(T)$. These matrices are square matrices if and only if the region T is balanced. According to Theorem 5.17, the determinants of $N(T)$ and $Z(T)$ have the same absolute value if T is a balanced triangular subregion, i.e., its punctures are upward-pointing triangles. However, we are not aware of any example, where the mentioned equality is not true. This raises the following question:

Question 12.1. Let $T \subset \mathcal{T}_d$ be any non-empty balanced subregion. Is then the equality

$$|\det Z(T)| = |\det N(T)|$$

always true?

An affirmative answer would extend the bijection between signed perfect matchings determined by T and signed families of non-intersecting lattice path in T from triangular subregions to arbitrary balanced subregions.

In this work we have focussed on studying the weak Lefschetz property of an Artinian monomial ideal $I \subset R = K[x, y, z]$ by establishing combinatorial interpretations of the multiplication maps $[A]_{d-1} \rightarrow [A]_d$ on $A = R/I$ by $\ell = x + y + z$. In order to study the strong Lefschetz property of A it would be desirable to extend our approach by finding combinatorial interpretations of the multiplication by powers of ℓ . This could also be an approach to determining the Jordan canonical form of the multiplication map by ℓ on A .

Quite generally, let A be a graded Artinian K -algebra with Hilbert function $h_A(j) = \dim_K[A]_j$. The multiplication on A by any linear form $0 \neq L \in A$ is a nilpotent map. Hence, the Jordan canonical form J_L of this multiplication map is given by a partition λ of $\dim_K A$. The parts of this partition are determined by the ranks of the multiplication by powers of L . More precisely, define $m = \max\{h_A(i) \mid i \in \mathbb{Z}\}$ and, for all $j = 1, \dots, m$,

$$\lambda_j = |\{i \mid h_A(i) \geq j\}|.$$

Then $\lambda = (\lambda_1, \dots, \lambda_m)$ is called the *expected partition*, determining the expected Jordan canonical form J_ℓ of the multiplication by a general linear form $\ell \in A$. This is closely related to the strong Lefschetz property. In fact, the algebra A has the strong Lefschetz property if and only if the Jordan canonical form of the multiplication on A by a general linear form is determined by the expected partition.

Since monomial complete intersections have the strong Lefschetz property in characteristic zero this observation allows us to determine the corresponding Jordan canonical form.

Example 12.2. Consider the algebra $A = R/I$, where $I = (x^3, y^4, z^5)$ and K is a field of characteristic zero. Its Hilbert function is positive from degree zero to degree nine. The corresponding values are

$$1, 3, 6, 9, 11, 11, 9, 6, 3, 1.$$

Hence, the Jordan canonical form of the multiplication on A by $\ell = x + y + z$ (or every general linear form) is given by the expected partition of 60,

$$\lambda = (10, 8, 8, 6, 6, 4, 4, 4, 2, 2).$$

The richness of the results on lozenge tilings, perfect matchings, and families of non-intersection lattice paths invites one to find higher-dimensional generalizations. MacMahon already considered the three-dimensional analogue of plane partitions in [42]. However,

numerical evidence suggests that there is no simple formula for enumerating such space partitions. Nevertheless, parts of the theory we developed here do extend to higher dimension. We hope that the connection to the weak Lefschetz property of quotients of polynomial rings in more than three variables can provide some guidance towards extending some of the beautiful classical combinatorial results.

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This work supersedes [16]. The authors thank a referee of [16] for very valuable comments.

APPENDIX A. COMPLETION OF THE PROOF OF THEOREM 7.11.

Adopt the notation introduced in Chapter 7. In order to prove Theorem 7.11 it remains to show the following result.

Proposition A.1. *Let $T = T_d(I)$ be a triangular region as in Assumption 7.1 with parameters b and $(h_1, d_1), \dots, (h_s, d_s)$, where $s \geq 1$ and d_2, \dots, d_{s-1} are all even. Define a , k , \mathbf{p} , and \mathbf{q} as introduced above Theorem 7.8.*

If d_1 is even, d_s is odd, and $h_s = 0$, then

$$\text{perm } Z(T) = 2^{m+n} \overline{P}_{\emptyset, \mathbf{q}} \left(\frac{a+k-1}{2} \right) \overline{P}_{\mathbf{q}, \emptyset} \left(\frac{a}{2} \right).$$

Proof. This follows as in [12, Part B, Section 3, Proof of Theorem 1.1] if we make a single adjustment.

In order to refrain from making copious new definitions, we assume the reader is familiar with the notation used in [12, Part B].

The penultimate sentence on page 123 of [12] ends with the phrase “while the one obtained from R^- after the same procedure is congruent to $R_{\mathbf{q}, \mathbf{l}^{(m)}}(a/2)$ ” (recall that we use \mathbf{p} in place of \mathbf{l} , as discussed above). This congruence hinges on the assumption given in the second sentence of the same paragraph: “Then the last vertebra below \mathcal{R} is in this case a triangular vertebra...” However, when $\mathbf{p} = \emptyset$, this assumption fails, and the stated congruence breaks down (see, e.g., Figure 7.2). Fortunately, we need simply replace $R_{\mathbf{q}, \mathbf{l}^{(m)}}(a/2) = R_{\mathbf{q}, \emptyset}(a/2)$ by $\overline{R}_{\mathbf{q}, \emptyset}(a/2)$ for the congruence to hold.

Set $R = R_{\mathbf{q}, \emptyset}(a/2)$ and $\overline{R} = \overline{R}_{\mathbf{q}, \emptyset}(a/2)$. Then we will show that R is \overline{R} with an extra column of triangles along the Northwestern boundary and an extra row of triangles along the Northern boundary, thus expanding the region considered. See Figure A.1 for an example of R and \overline{R} .

When expanding R and \overline{R} to the symmetric regions of which they are part, we need only consider those triangles that are below the horizontal line that goes through the origin O since $\mathbf{p} = \emptyset$. Further, the first selected bump is q_1 , and as d_s is odd, we have $q_1 = 1 + \lfloor \frac{d_s}{2} \rfloor = \frac{1}{2}(d_s + 1)$.

When expanding R to the symmetric region T which it is part of, we notice that the top of the q_1 bump is d_s triangles below O . Thus the Northwest path coming from the top of the bump joins the Northwest edge of the bump, and this ray extends to the horizontal ray to the West of O . This creates a Northwest edge that is $d_s + 1$ units long. Hence the lowest axial puncture of T has side-length $d_s + 1$. In particular, $T = T_{d+1}(\hat{I})$, where \hat{I} is generated

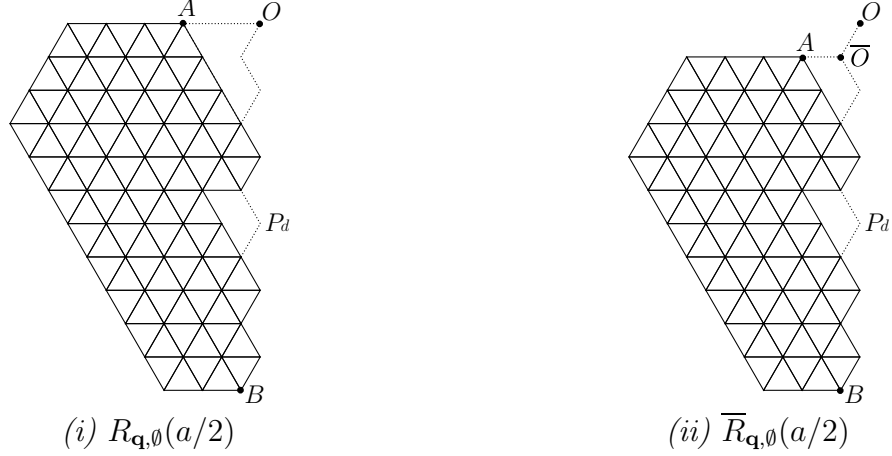


FIGURE A.1. An example of the distinction between $R = R_{\mathbf{q}, \emptyset}(a/2)$ and $\overline{R} = \overline{R}_{\mathbf{q}, \emptyset}(a/2)$, where $\mathbf{q} = \{2, 4, 5\}$ and $a = 4$. The region R is the same as [12, Part B, Figure 2.1(c)], and \overline{R} is very similar to [12, Part B, Figure 2.2(c)].

by the same generators as I , but with the following modifications: all generators divisible by x are multiplied by x and the pure powers of y and z are multiplied by y and z , respectively. See Figure A.2(i).

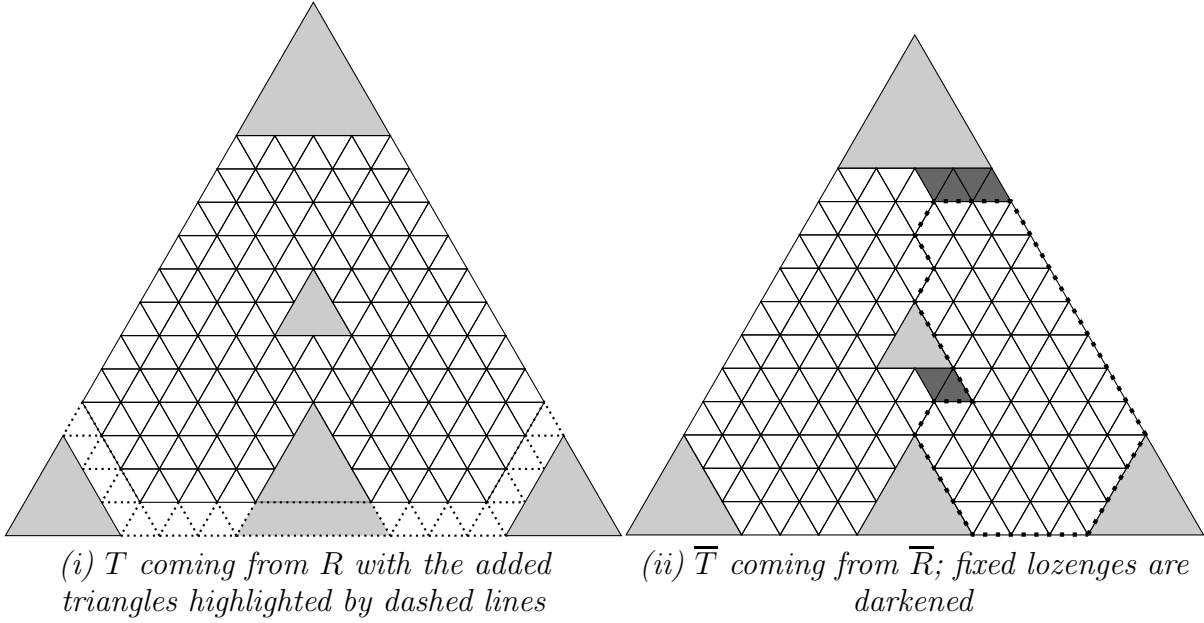


FIGURE A.2. Let $\mathbf{q} = \{2, 4, 5\}$ and $a = 4$, as in Figure A.1. Then $T = T_{16}(x^{12}, y^{13}, z^{13}, x^6 y^4 z^4, y^6 z^6)$ and $\overline{T} = T_{15}(x^{11}, y^{12}, z^{12}, x^5 y^4 z^4, y^6 z^6)$.

On the other hand, when expanding \overline{R} to the symmetric region \overline{T} which it is part of, we notice that the top of the q_1 bump is $d_s - 1$ triangles below \overline{O} . Thus the lowest axial puncture of T has side-length d_s . Hence $\overline{T} = T_d(I)$, as desired. See Figure A.2(ii).

The claim now follows by using the arguments in [12, Part B] and replacing the region R by the region \overline{R} . \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556, USA
E-mail address: `dcook8@nd.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, 715 PATTERSON OFFICE TOWER, LEX-
 INGTON, KY 40506-0027, USA
E-mail address: `uwe.nagel@uky.edu`